Material Strength: A Rational Nonequilibrium Energy Model for Complex Loadings

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The failure of materials with some sort of loading is a well-known natural phenomenon, and the reliable prediction of the failure of materials is the most important issue for many different kinds of engineering materials based on safety considerations. Classical strength theories with complex loadings are based on some sort of postulations or assumptions, and they are intrinsically empirical criteria. Due to their simplicity, classical strength theories are still widely used in engineering, and they are very easy to incorporate into any finite element code. Recently, a new methodology was proposed by the author. Instead of establishing empirical models, the material failure process was modeled as a nonequilibrium process. Then, the strength criterion was established with the rational stability analysis for the failure process. In this study, the author tried to use this idea to develop a rational thermodynamic strength theory and to make the theory easy to use in engineering, similar to the classical strength criteria. It was found that the predictions of the rational energy strength theory were very reasonable compared to the experimental data even if no postulation was taken. Through the analysis, it seemed that the strength problem could be efficiently tackled using the rational nonequilibrium energy model instead of using some sort of empirical assumptions or models. [DOI: 10.1115/1.4048988]

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1 Introduction

The mechanical strength of materials is a key issue in many fields. From the point of view of scientists and engineers in the field of mechanical engineering, the strength criterion of a material refers to the critical conditions for a material specimen to fail during any complex loadings, assuming that one knows some critical strength values of the material for some simple loading, such as uniaxial tensile strength and simple shear strength. This has been a long-standing problem lasting for hundreds of years, and the famous strength criteria and their variations have been a standard content in the textbooks of the mechanics of materials. However, intrinsically, these criteria are empirical, and to make a safe design for any components and structures in engineering, one has to adopt some very big safety factors to overcome the uncertainty in the prediction.

Material failure under external loading is a very complicated phenomenon. For different materials, a failure mechanism is intrinsically different, and for the same material with different loadings, a failure mechanism may be very different. The failure process may involve the creation, accumulation, and growth of the various defects such as dislocations, voids, and microcracks. To satisfy engineering demands, researchers have developed strength criteria based mainly on two approaches. One approach is to develop phenomenological models to achieve a more accurate prediction of material strength [1]. The other approach is to use so-called micromechanics models based on the failure mechanism to develop the prediction models of the failure of a material [2–4]. Recently, Wang [5] modeled the failure process of materials as a nonequilibrium thermodynamic process and developed a rational energy methodology to predict the failure of a material. In fact, the Griffith energy release rate theory, which is used to predict crack propagation [6], and the field theoretic formulation, which is used to predict the landslip property along some faults in earthquake theory [7–9], are energy theories with the same origin. Recently, many theoretical works have been conducted based on first-principle calculations for the theoretical strength of diamond and other ideal materials [10–12]. In addition, it can be shown that the energy approach developed by the author can provide correct predictions.

The classical strength theory is very easy to use for engineers and other users. For this theory, the strength data for different materials are obtained using the standard specimens for simple loading conditions, and the data are then collected into some engineering handbooks. This theory is also very convenient to incorporate into finite element method codes. Thus, users can evaluate the stress distribution and the strength at the same time. In this study, the author attempted to develop an easy-to-use method for evaluating material strength based on the nonequilibrium thermodynamic scheme. In the future, it is hoped that it will be shown that the energy approach could also be incorporated easily with some numerical and experimental methods to solve the strength problem of materials.

2 Nonequilibrium Energy Model

When considering a material specimen with some sort of external loading, when the load is small, the specimen will deform elastically without any energy dissipation or entropy production. By increasing the external load, some sort of defects will be created or activated in the specimen, thus inducing entropy production. Based on nonequilibrium thermodynamics [13], it is known that the changing rate of the microstructural defect parameter can be defined as the "flow" of the process, and the conjugate driving force is the derivative of the entropy production with respect to

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the corresponding defect parameter. In other words, if the defect parameter can be denoted as a_i, the conjugate driving force can be denoted as F_i. The first law of thermodynamics can be expressed as follows:

$$\delta Q = dU - \delta W, \quad \delta W = \iint_{\Gamma} \Pi_i \delta u_i ds \tag{1}$$

where dU is the internal energy increment. For static problems, dU might refer to the elastic strain energy. δW is the environment work on the system, Π_i, u_i, and Γ refer the force and displacements components along the boundary Γ, and δQ is the heat supplied to the system.

For a nonequilibrium process, the state function, the entropy S, is introduced, and it can be divided into the exchange entropy due to the exchange with the environment and irreversible entropy production, as follows:

$$dS = dS_r + dS_i; \quad dS_r = \frac{\delta Q}{T}; \quad dS_i \geq 0 \tag{2}$$

which is also a state of the second law of thermodynamics. By combining Eqs. (1) and (2), one can derive

$$\begin{aligned} dS_i &= dS - dS_r = dS - \frac{\delta Q}{T} = dS - \frac{1}{T}(dU - \delta W) \\ &= -\frac{1}{T}(dU - \delta W - TdS) \end{aligned} \tag{3}$$

By introducing the definition of the Gibbs free energy, one can write

$$G = U - W - ST \tag{4}$$

Its variation can be written in the following form:

$$\begin{aligned} dG &= dU - \int_{\Gamma} \delta(\Pi_i u_i) ds - d(ST) \\ &= dU - \int_{\Gamma} \Pi_i \delta u_i ds - \int_{\Gamma} u_i \delta \Pi_i ds - TdS - SdT \end{aligned} \tag{5}$$

By combining the earlier equations with Eq. (3), one can establish

$$dS_i = -\frac{1}{T}(dG + \int_{\Gamma} u_i \delta \Pi_i ds - SdT) = -\frac{1}{T} dG; \quad \delta \Pi_i = 0, \ dT = 0 \tag{6}$$

It is very clear that in deriving Eq. (6), it is assumed that for the condition of keeping the external loading and temperature constant, the entropy production can be expressed as the decrease of the Gibbs free energy.

According to nonequilibrium thermodynamics, if the entropy production is due to defect creation and evolution, the thermodynamic driving force can be expressed in the following form:

$$F_i = \frac{\partial S_i}{\partial a_i} = -\frac{1}{T}\frac{\partial G}{\partial a_i} \tag{7}$$

Its conjugate "flow" can be expressed as follows:

$$\frac{da_i}{dt} = \lambda F_i = -\frac{\lambda}{T}\frac{\partial G}{\partial a_i} \tag{8}$$

In fact, for a general nonequilibrium process not very far from the equilibrium state, it is a well-known fact that the "flow" is linearly related to its energy driving force [13]. At the least, it is a well-known assumption. Thus, one could establish the evolution equation of the damaged microstructures of a material under external loading. To determine the strength of a material specimen, generally speaking, it is not necessary to derive the solution of the evolution equation. If one can establish the stability condition of the evolution equation, the critical condition gives the strength of the material specimen. This is the basic idea for determining how to evaluate the material strength. In other words, based on the failure mechanism, by choosing a proper parameter to represent the "flow" of the system, one can establish the nonequilibrium energy evolution equation, from which one can obtain the critical failure conditions of the materials. Compared with the classical strength theory, the energy criterion is a global, rational criterion. In the following sections, we use some examples to show how to use the formalism for material failure problems.

In fact, if one is only interested in establishing the critical criterion for the material failure, one only needs to consider the stability condition for an equilibrium state with loading, i.e., using the second-order variation of the Gibbs free energy at given load and temperature. If this variation is smaller or equal to zero, it means that the equilibrium state is not stable for any sort of perturbation. Thus, one can establish the strength criterion by setting the second-order variation of the Gibbs free energy equal to zero under uniaxial loading or by analyzing the condition for which the quadratic form obtained from the second-order derivatives of the Gibbs free energy is equal to zero under complex loadings.

3 Energy Expressions Based on the Eigenstrain

The kernel of the energy methodology for predicting strength is to find the Gibbs free energy for a material with loading. In this study, the concept of "Eigenstrain" was used to help us to derive the expressions of the Gibbs free energy, as follows.

Eigenstrain is a generic name given by Mura [14] for nonelastic strains for thermal expansion, phase transformation, and misfit strains. In his book, it is very clear that many kinds of defects, such as voids, dislocations, and cracks, could be modeled by the eigenstrains. Thus, both the Gibbs free energy and the dissipation energy can be expressed as a function of the eigenstrains.

Considering a homogeneous material under the action of some sort of external loading, some defects may be created or activated, and one may say that some distribution of the eigenstrains has been produced in the material specimen. For example, one dislocation in a homogeneous material can be simulated by the eigenstrain distribution as follows:

$$\varepsilon_{ij}^T(\mathbf{x}) = -\frac{1}{2}(b_i n_j + b_j n_i)\delta(\mathbf{S} - \mathbf{x}) \tag{9}$$

where $\delta(\mathbf{S} - \mathbf{x})$ is the one-dimensional Dirac function in the normal direction of $\mathbf{S}$, being unbounded when $\mathbf{x}$ is equal to $\mathbf{S}$ and zero otherwise, and $\mathbf{b}$ and $\mathbf{n}$ are the burgers vector and the normal vector, respectively, toward S^-. A mode-I penny-shaped crack ($a_1 = a_2 = a$; $a_3 = 0$) can be simulated by the eigenstrain:

$$\varepsilon^T(\mathbf{x}) = \lim_{a_3 \to 0} a_3 \varepsilon_{33}^T = \text{finite} \tag{10}$$

The other components are zero. With the creation of the distribution of the eigenstrains, the elastic Gibbs free energy of the system will change, and some sort of energy dissipation cannot be avoided. By introducing the concept of the eigenstrain, one can easily derive the free energy as follows.

Considering a material specimen under external loading σ_{ij}^0, the elastic Gibbs free energy can be written in the following form:

$$W_0 = \frac{1}{2}\iiint_D \sigma_{ij}^0 \varepsilon_{ij}^0 dv - \iint_{\Gamma} F_i^0 u_i^0 ds \tag{11}$$

When the load reaches some critical value, the irreversible defects will be created or activated. Thus, the eigenstrain $\varepsilon_{ij}^T(\mathbf{x})$ becomes nonzero, and the elastic Gibbs free energy can be written as follows:

$$\begin{aligned} W &= \frac{1}{2}\iiint_D (\sigma_{ij}^0 + \sigma_{ij})(\varepsilon_{ij}^0 + \varepsilon_{ij} - \varepsilon_{ij}^T) dv - \iint_{\Gamma} F_i^0 (u_i^0 + u_i) ds \\ &= \frac{1}{2}\iiint_D (\sigma_{ij}^0 + \sigma_{ij})(\varepsilon_{ij}^0 + \varepsilon_{ij} - \varepsilon_{ij}^T) dv - \iint_{\Gamma} \sigma_{ij}^0 n_j (u_i^0 + u_i) ds \end{aligned}, \tag{12}$$

where σ_{ij}, ε_{ij}, and u_i are the perturbations of the stress, strain, and displacement due to the eigenstrains. Using Green's function, one can obtain [14]

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl}\iiint_D e_{sth}e_{lnh}C_{pqmn}T_{kp,qt}(\mathbf{x}-\mathbf{x}')\varepsilon^T_{sm}dv' \qquad e_{sth}e_{lnh} = \delta_{sl}\delta_{tn} - \delta_{sn}\delta_{tl} \tag{13}$$

where $T_{kp,qt}(\mathbf{x}-\mathbf{x}')$ is the second-order derivative of Green's function, C_{ijkl} is the elastic moduli tensor, and δ_{sl} is the Kronecker delta. The elastic Gibbs free energy change due to the occurrence of the eigenstrain can be derived as follows:

$$\Delta W = W - W_0 = -\frac{1}{2}\iiint_D (2\sigma^0_{ij} + \sigma_{ij})\varepsilon^T_{ij}dv \tag{14}$$

In deriving Eq. (14), the following conditions were used:

$$\iiint_D \sigma^0_{ij}(\varepsilon_{ij} - \varepsilon^p_{ij})dv = \iiint_D \sigma_{ij}(\varepsilon^0_{ij} + \varepsilon_{ij})dv = 0 \tag{15}$$

Accompanying the creation of the eigenstrain, the resistant force will do the work and increase the energy dissipation and the free energy of a system. It is then necessary to determine how the dissipation energy depends on the eigenstrain. This is the most crucial problem. For a crack propagation problem, one can assume that the surface energy plays the role of energy dissipation. Since the purpose is to establish a general methodology to evaluate the material strength, a general form of energy dissipation is used based on the Landau–Lifshitz free energy expansion [15]. The total resistant free energy can be written in the following form:

$$\Lambda = \iiint_D \Delta[\varepsilon^T_{kl}(\mathbf{x})]dv \tag{16}$$

If the material specimen being considered is isotropic, since the eigenstrain is a second-order symmetric tensor, only three invariant scalars can form. One form can be expressed as follows [16]:

$$\begin{aligned} I_1 &= \varepsilon^T_{ll} = \varepsilon^T_{11} + \varepsilon^T_{22} + \varepsilon^T_{33} \\ I_2 &= (\varepsilon^T_{ik})^2 = (\varepsilon^T_{11})^2 + (\varepsilon^T_{22})^2 + (\varepsilon^T_{33})^2 + 2(\varepsilon^T_{12})^2 + 2(\varepsilon^T_{13})^2 + 2(\varepsilon^T_{23})^2 \\ I_3 &= \varepsilon^T_{ik}\varepsilon^T_{il}\varepsilon^T_{kl} = \varepsilon^T_{11}[(\varepsilon^T_{11})^2 + (\varepsilon^T_{12})^2 + (\varepsilon^T_{13})^2] + \varepsilon^T_{22}[(\varepsilon^T_{12})^2 + (\varepsilon^T_{22})^2 + (\varepsilon^T_{23})^2] + \varepsilon^T_{33}[(\varepsilon^T_{13})^2 + (\varepsilon^T_{23})^2 + (\varepsilon^T_{33})^2] + 2\varepsilon^T_{12}(\varepsilon^T_{11}\varepsilon^T_{12} + \varepsilon^T_{12}\varepsilon^T_{22} + \varepsilon^T_{13}\varepsilon^T_{23}) \\ &\quad + 2\varepsilon^T_{13}(\varepsilon^T_{11}\varepsilon^T_{13} + \varepsilon^T_{12}\varepsilon^T_{23} + \varepsilon^T_{13}\varepsilon^T_{33}) + 2\varepsilon^T_{23}(\varepsilon^T_{12}\varepsilon^T_{13} + \varepsilon^T_{22}\varepsilon^T_{23} + \varepsilon^T_{23}\varepsilon^T_{33}) \end{aligned} \tag{17}$$

By assuming that the eigenstrains are small quantities, the dissipation energy can be expanded in the form of

$$\Delta(\varepsilon^T_{kl}) = AI_1 + BI_1^2 + CI_2 + DI_1^3 + EI_1I_2 + FI_3 + \cdots \tag{18}$$

One should notice that for an isotropic material and a symmetrical tensor of rank two, only one linear scalar, two quadratic scalars, and three third-order scalars can be formed. In the following expressions, the dissipation energy density expansion up to the third order is used. In fact, it is well known that the deformation characteristics for the tension and compression may be different. Therefore, the constants should be determined according to different loading conditions. In fact, the idea is that for any kind of material, if the six constants in the expansion can be determined through some simple testing such as uniaxial loading, the strength curve for complex loading can be derived similar to the classic strength theory. Of course, for some special materials, the dissipation energy density may not be expanded as shown in Eq. (18), so one needs to find its expression in some other way. This reflects the resistance against failure of the materials. Nonetheless, for most isotropic metal materials, such an expansion should be quite versatile.

The substitution of Eqs. (14) and (16) into Eq. (8) gives the evolution equation of the eigenstrains:

$$\begin{aligned} \frac{d\varepsilon^T_{ij}}{dt} &= -\lambda\frac{\delta G}{\delta\varepsilon^T_{ij}} = -\lambda\frac{\delta(\Delta W + \Lambda)}{\delta\varepsilon^T_{ij}} = \lambda\left(\sigma^0_{ij} + \sigma_{ij} - \frac{d\Delta}{d\varepsilon^T_{ij}}\right) \\ &= \lambda\left[\sigma^0_{ij} - \frac{d\Delta}{d\varepsilon^T_{ij}} + C_{ijkl}e_{sth}e_{lnh}C_{pqmn}\iiint_D T_{kp,qt}(\mathbf{x}-\mathbf{x}')\varepsilon^T_{sm}dv'\right] \end{aligned} \tag{19}$$

By setting the evolution velocity to be zero, one obtains

$$\sigma^0_{ij} + \sigma_{ij} - \frac{d\Delta}{d\varepsilon^T_{ij}} = \sigma^0_{ij} - \frac{d\Delta}{d\varepsilon^T_{ij}} + C_{ijkl}e_{sth}e_{lnh}C_{pqmn}\iiint_D T_{kp,qt}(\mathbf{x}-\mathbf{x}')\varepsilon^T_{sm}dv' = 0 \tag{20}$$

where

$$\begin{aligned} \frac{d\Delta}{d\varepsilon^T_{mn}} &= \Omega_{mn} \\ &= A\delta_{mn} + 2BI_1\frac{dI_1}{d\varepsilon^T_{mn}} + C\frac{dI_2}{d\varepsilon^T_{mn}} + 3DI_1^2\frac{dI_1}{d\varepsilon^T_{mn}} \\ &\quad + E\left(I_1\frac{dI_2}{d\varepsilon^T_{mn}} + I_2\frac{dI_1}{d\varepsilon^T_{mn}}\right) + F\frac{dI_3}{d\varepsilon^T_{mn}} + \cdots \end{aligned} \tag{21}$$

and

$$\begin{aligned} \frac{dI_1}{d\varepsilon^T_{mn}} &= \delta_{mn} \\ \frac{dI_2}{d\varepsilon^T_{mn}} &= 2\varepsilon^T_{mn} \\ \frac{dI_3}{d\varepsilon^T_{mn}} &= \varepsilon^T_{ml}\varepsilon^T_{nl} + \varepsilon^T_{mk}\varepsilon^T_{kn} + \varepsilon^T_{im}\varepsilon^T_{in} \end{aligned} \tag{22}$$

It is very clear that Eq. (20) is an integral equation for the eigenstrain distribution. To simplify the problem, one can assume that the eigenstrain is uniform in the specimen. Thus, the following

expression can be derived:

$$\sigma^0_{mn} = A\delta_{mn} + 2BI_1\frac{dI_1}{d\varepsilon^T_{mn}} + C\frac{dI_2}{d\varepsilon^T_{mn}} + 3DI_1^2\frac{dI_1}{d\varepsilon^T_{mn}} + E\left(I_1\frac{dI_2}{d\varepsilon^T_{mn}} + I_2\frac{dI_1}{d\varepsilon^T_{mn}}\right) + F\frac{dI_3}{d\varepsilon^T_{mn}} + \cdots \quad (23)$$

Equation (23) presents the relationship between the applied load and the eigenstrain. The total strain of the material is the elastic strain added to the eigenstrain because the eigenstrain is assumed to be uniform. Thus, one can derive the relationship between the applied stresses with the total strain. If one can obtain the stress–strain relationship through a simple loading test, one can determine the material parameters A, B, C, D, E, and F. It should be also noticed that the stress perturbation due to the eigenstrain is zero for the uniform distribution of the eigenstrain in a specimen, and the parameter A corresponds to the yielding point of a material.

The question then becomes how to determine the strength value, especially when the specimen is under complex loading. The Gibbs free energy G is obtained using the concept of the eigenstrain. In fact, Eq. (23) is derived with $dG/d\varepsilon^T_{ij} = 0$. For some equilibrium state, if the eigenstrain has some small perturbation $\Delta\varepsilon^T_{ij}$, the free energy can be expressed in the form of

$$G(\varepsilon^T_{ij} + \Delta\varepsilon^T_{ij}) = G(\varepsilon^T_{ij}) + \frac{dG}{d\varepsilon^T_{ij}}\Delta\varepsilon^T_{ij} + \frac{1}{2}\frac{d^2G}{d\varepsilon^T_{ij}d\varepsilon^T_{\alpha\beta}}\Delta\varepsilon^T_{ij}\Delta\varepsilon^T_{\alpha\beta} + \cdots = G(\varepsilon^T_{ij}) + \frac{1}{2}\frac{d^2G}{d\varepsilon^T_{ij}d\varepsilon^T_{\alpha\beta}}\Delta\varepsilon^T_{ij}\Delta\varepsilon^T_{\alpha\beta} + \cdots \quad (24)$$

where

$$\begin{aligned}\Pi_{ij\alpha\beta} &= \frac{\partial^2 G}{\partial\varepsilon^T_{ij}\partial\varepsilon^T_{\alpha\beta}} = \frac{\partial^2\Delta}{\partial\varepsilon^T_{ij}\partial\varepsilon^T_{\alpha\beta}} \\ &= 2A\frac{\partial I_1}{\partial\varepsilon^T_{ij}}\frac{\partial I_1}{\partial\varepsilon^T_{\alpha\beta}} + B\frac{\partial^2 I_2}{\partial\varepsilon^T_{ij}\partial\varepsilon^T_{\alpha\beta}} + 12CI_1^2\frac{\partial I_1}{\partial\varepsilon^T_{ij}}\frac{\partial I_1}{\partial\varepsilon^T_{\alpha\beta}} \\ &+ 2D\left(\frac{\partial I_2}{\partial\varepsilon^T_{\alpha\beta}}\frac{\partial I_2}{\partial\varepsilon^T_{ij}} + I_2\frac{\partial^2 I_2}{\partial\varepsilon^T_{ij}\partial\varepsilon^T_{\alpha\beta}}\right) \\ &+ E\left(\frac{\partial I_1}{\partial\varepsilon^T_{\alpha\beta}}\frac{\partial I_3}{\partial\varepsilon^T_{ij}} + \frac{\partial I_3}{\partial\varepsilon^T_{\alpha\beta}}\frac{\partial I_1}{\partial\varepsilon^T_{ij}} + I_1\frac{\partial^2 I_3}{\partial\varepsilon^T_{ij}\partial\varepsilon^T_{\alpha\beta}}\right) \\ &+ F\left[2\left(\frac{\partial I_1}{\partial\varepsilon^T_{\alpha\beta}}I_2 + I_1\frac{\partial I_2}{\partial\varepsilon^T_{\alpha\beta}}\right)\frac{\partial I_1}{\partial\varepsilon^T_{ij}} + 2I_1\frac{\partial I_1}{\partial\varepsilon^T_{\alpha\beta}}\frac{\partial I_2}{\partial\varepsilon^T_{ij}} + I_1^2\frac{\partial^2 I_2}{\partial\varepsilon^T_{ij}\partial\varepsilon^T_{\alpha\beta}}\right]\end{aligned} \quad (25)$$

It is very clear that if the quadratic form in Eq. (24) is larger than zero, any perturbation will make the free energy increase, and the state will be stable. In contrast, if the quadratic form is smaller than zero for some perturbation, the state will not be stable. Thus, one can establish that the critical condition for the material failure is that the quadratic form is zero. Substitution into Eq. (23) gives the critical condition of applied loadings for material failure.

4 Experimental Verification: From Simple Loading Data to Complex Loading Prediction

To explain the basic idea of this study, as described in this section, some materials were chosen as examples. Based on the stress–strain relationship in simple uniaxial loading, the material parameters were determined. Then, by using Eq. (25), one could determine the strength criterion of the material with complex loadings. The predictions were verified with experimental measurements. For the uniaxial tensile loading, using Eq. (23), one could establish the equation for the eigenstrain components, or the plastic strains with the applied loading, from which one could determine the material constants. It was very clear that to determine all six parameters, one needed at least two sets of the experimental results of both uniaxial tensile and shear loading.

The experimental data obtained by Chino et al. [17] for AZ31 Mg alloy were used. They obtained the data for four types of specimens. The detailed information is referred to in their publication. According to their work, it was clear that the eigenstrains could be treated as the plastic strains, and one could assume that the plastic deformation would not induce the volume change, i.e.,

$$I_1 = \varepsilon^T_{ll} = \varepsilon^T_{11} + \varepsilon^T_{22} + \varepsilon^T_{33} = 0 \quad (26)$$

4.1 Uniaxial Tensile Loading. In this case, $\varepsilon^T_{22} = \varepsilon^T_{33} = -1/2\varepsilon^T_{11}$, and Eq. (23) became

$$\begin{aligned}\sigma^0_{11} &= A + C\frac{dI_2}{d\varepsilon^T_{11}} + EI_2\frac{dI_1}{d\varepsilon^T_{11}} + F\frac{dI_3}{d\varepsilon^T_{11}} + \cdots \\ &= A + 2C\varepsilon^T_{11} + EI_2 + 3F(\varepsilon^T_{11})^2 \\ &= A + 2C\varepsilon^T_{11} + \frac{3}{2}E(\varepsilon^T_{11})^2 + 3F(\varepsilon^T_{11})^2 \\ &= A + 2C\varepsilon^T_{11} + \left(\frac{3}{2}E + 3F\right)(\varepsilon^T_{11})^2 \\ \sigma^0_{12} &= C\frac{dI_2}{d\varepsilon^T_{12}} + EI_2\frac{dI_1}{d\varepsilon^T_{12}} + F\frac{dI_3}{d\varepsilon^T_{12}} + \cdots \\ &= 2C\varepsilon^T_{12} + 3F(\varepsilon^T_{11} + \varepsilon^T_{22})\varepsilon^T_{12} + 3F\varepsilon^T_{13}\varepsilon^T_{23}\end{aligned} \quad (27)$$

Since the material considered was not superplastic, and it was not unreasonable to assume that the normal plastic strain was not coupled with the shear strain, $F = 0$. Thus, the first equation in Eq. (27) became

$$\sigma^0_{11} = A + 2C\varepsilon^T_{11} + \frac{3}{2}E(\varepsilon^T_{11})^2 \quad (28)$$

Using the stress–strain data for the uniaxial loading (Fig. 4 in Ref. [17]), one could determine the parameters A, C, and E of the material, which were used to predict the strength contour of the material under biaxial loadings. In fact, for uniaxial tensile loading, the position of the maximum stress was given by

$$\frac{\partial\sigma^0_{11}}{\partial\varepsilon^T_{11}} = 0, \quad (\varepsilon^T_{11})_{\max} = -\frac{2}{3}\cdot\frac{C}{E} \quad (29)$$

The parameters for the four types of specimens were obtained with the fitting scheme, which is presented in Table 1.

4.2 Strength With the Biaxial Loadings. For the purpose of predicting the strength contour curve for the biaxial loading, the first

Table 1 Fitting parameters for the four types of AZ31 alloy specimens

	A	C	E	$-2C/3E$
Specimen A	247.5	409.5	−1716.9	0.158
Specimen B	170.1	647.3	−2457.6	0.176
Specimen C	200.3	672.2	−3489.9	0.128
Specimen D	142.1	980.4	−5242.7	0.125

two equations in Eq. (23) became

$$\begin{aligned}\sigma_{11}^{0} &= A + C\frac{\partial I_2}{\partial \varepsilon_{11}^{T}} + EI_2\frac{\partial I_1}{\partial \varepsilon_{11}^{T}} \\ &= A + 2C\varepsilon_{11}^{T} + E[(\varepsilon_{11}^{T})^2 + (\varepsilon_{22}^{T})^2 + (\varepsilon_{11}^{T} + \varepsilon_{22}^{T})^2], \\ \sigma_{22}^{0} &= A + C\frac{\partial I_2}{\partial \varepsilon_{22}^{T}} + EI_2\frac{\partial I_1}{\partial \varepsilon_{22}^{T}} \\ &= A + 2C\varepsilon_{22}^{T} + E[(\varepsilon_{11}^{T})^2 + (\varepsilon_{22}^{T})^2 + (\varepsilon_{11}^{T} + \varepsilon_{22}^{T})^2]\end{aligned} \tag{30}$$

In addition, the components for the second derivation of the free energy were given by Eq. (25), as follows:

$$\begin{aligned}\frac{\partial^2 \Pi}{\partial (\varepsilon_{11}^{T})^2} &= 2B + 4C + 4E(2\varepsilon_{11}^{T} + \varepsilon_{22}^{T}), \\ \frac{\partial^2 \Pi}{\partial (\varepsilon_{22}^{T})^2} &= 2B + 4C + 4E(2\varepsilon_{22}^{T} + \varepsilon_{11}^{T}), \\ \frac{\partial^2 \Pi}{\partial \varepsilon_{11}^{T}\partial \varepsilon_{22}^{T}} &= 2B + 2C + 6E(\varepsilon_{11}^{T} + \varepsilon_{22}^{T}), \\ \frac{\partial^2 \Pi}{\partial \varepsilon_{22}^{T}\partial \varepsilon_{11}^{T}} &= 2B + 2C + 6E(\varepsilon_{11}^{T} + \varepsilon_{22}^{T})\end{aligned} \tag{31}$$

From Eq. (23), one could find that even if the material could not keep its volume unchanged in the deformation process, the contribution of

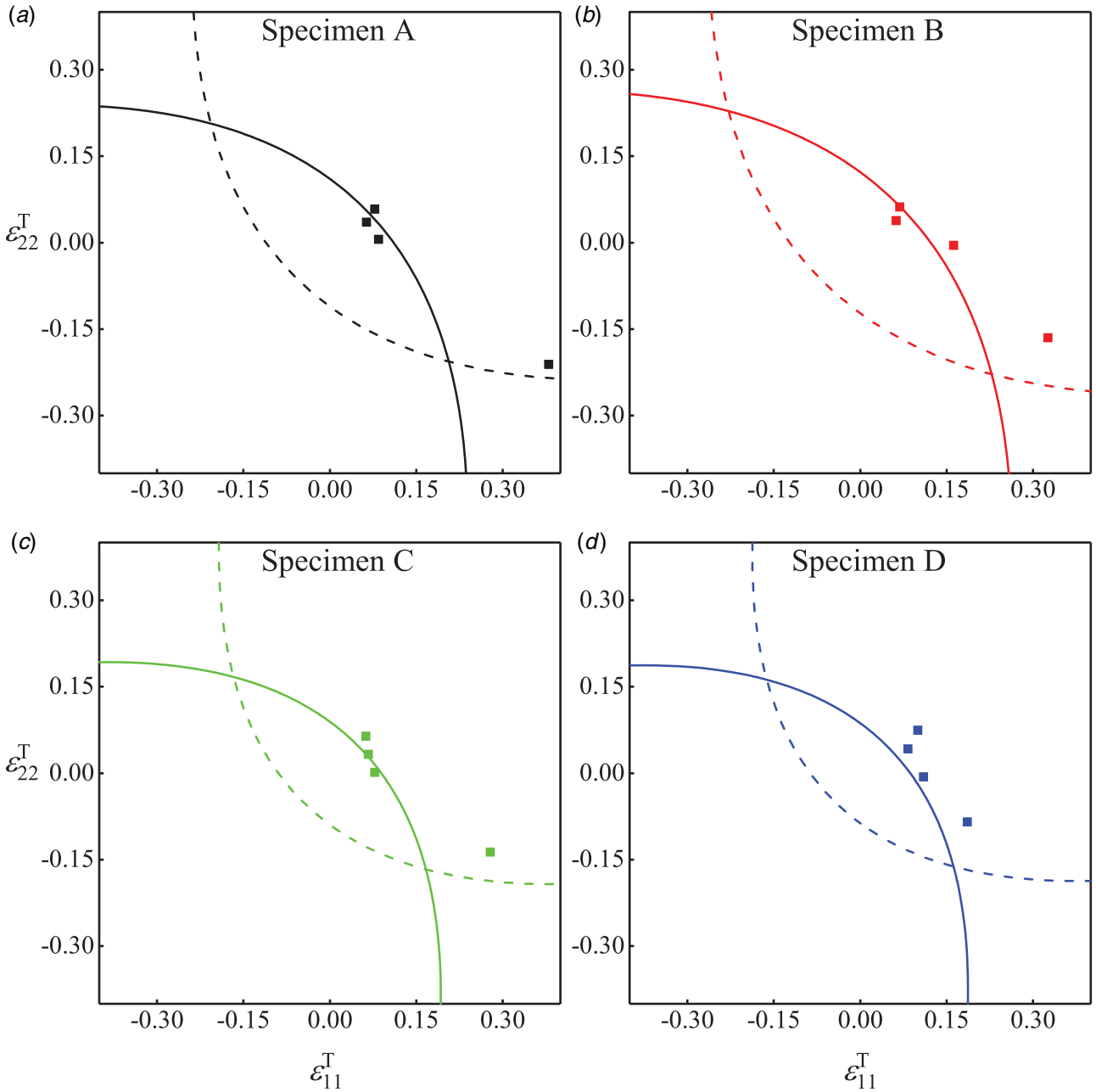

Fig. 1 The strength contour of the four types of specimens in plastic strain space. The solid (dashed) curves were given by Eq. (31) with the parameters *A*, *C*, and *E* obtained for the uniaxial tensile (compressive) loading. The dots represent the experimental data.

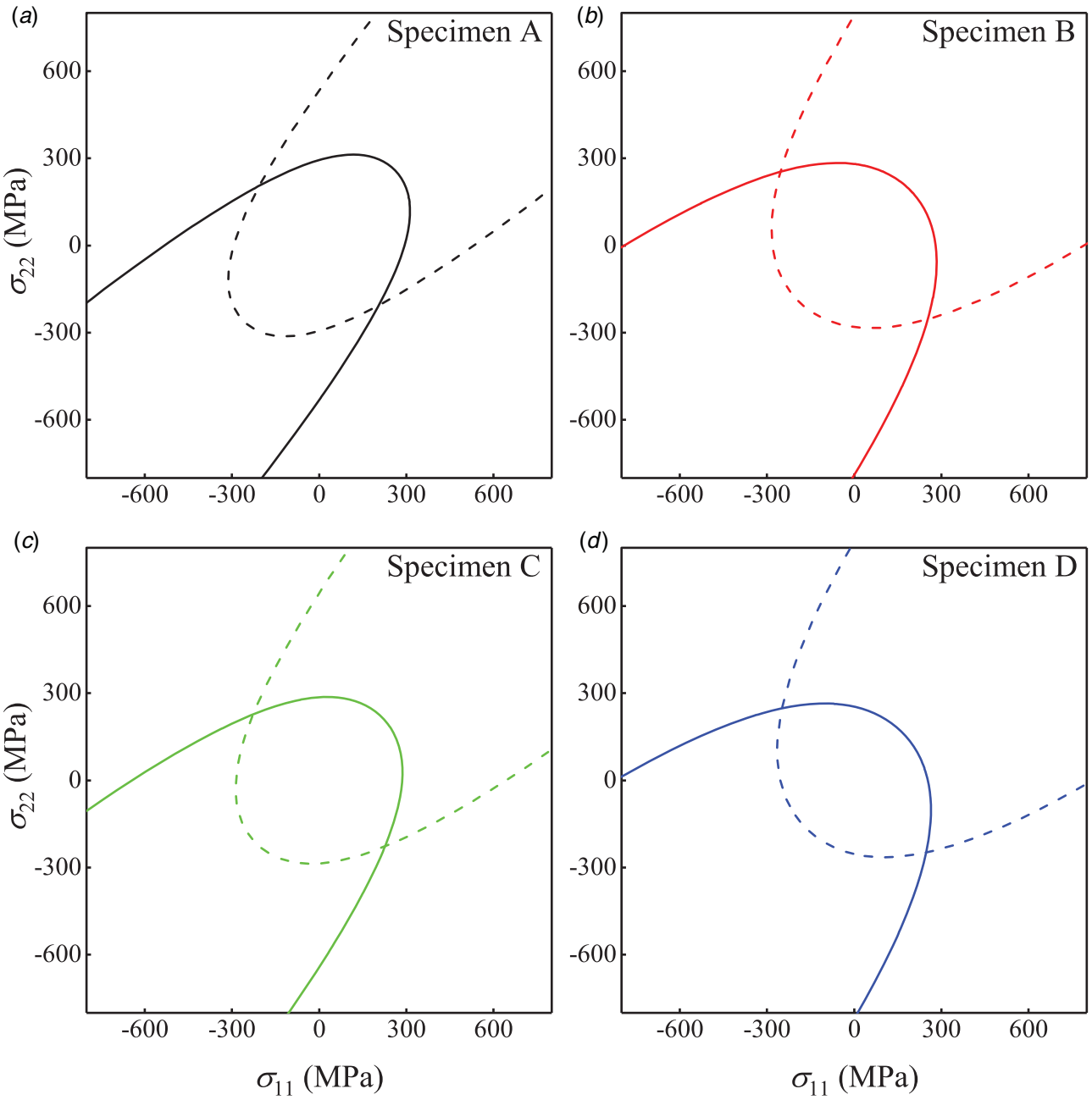

Fig. 2 The strength contours for the four types of specimens in stress space. The solid (dashed) curves were given by Eq. (30) with the parameters *A*, *C*, and *E* obtained for the uniaxial tensile (compressive) loading.

parameter B could also be adopted into C. Therefore, one could take $B = 0$. As discussed in Sec. 4.1, if the quadratic form given by Eq. (31) was larger than zero, any perturbation would make the free energy increase, and the state was stable, whereas if the quadratic form was smaller than zero for some perturbation, the state was not stable. To make the matrix of Eq. (31) positive, one could set

$$C + E(2\varepsilon_{11}^T + \varepsilon_{22}^T) > 0,$$
$$4[C + E(2\varepsilon_{11}^T + \varepsilon_{22}^T)][C + E(2\varepsilon_{22}^T + \varepsilon_{11}^T)] - [C + 3E(\varepsilon_{11}^T + \varepsilon_{22}^T)]^2$$
$$= E^2[(\varepsilon_{11}^T)^2 + (\varepsilon_{22}^T)^2] + 2E^2\varepsilon_{11}^T\varepsilon_{22}^T + 6CE(\varepsilon_{11}^T + \varepsilon_{22}^T) + 3C^2 > 0 \quad (32)$$

which gave the critical conditions for the material failure for the biaxial loading. It was obvious that if the second equation in Eq. (32) was satisfied, the first equation was automatically satisfied.

All of the experimental data listed in Table 1 were obtained with uniaxial tensile loading. If it was assumed that the stress–strain relationship of the materials considered here followed the same trend for both tensile and compressive loadings, then for the uniaxial compressive loading, the signs of the material parameters A and E should have changed, whereas their values and the parameter C should have stayed unchanged.

For the biaxial loading of the specimens, the strength contour lines predicted for the four types of specimens by the second equation in Eq. (32) in plastic strain space are shown in Fig. 1, along with the experimental data obtained by Chino et al. [17], where the elastic deformation was neglected. One should notice that no other postulations or assumptions were taken, except that the dissipated energy could be expanded into the power form, Eq. (18), and the predictions were quite reasonable. To obtain the strength contour lines in stress space, condition (32) was substituted into Eq. (30) and is shown Fig. 2. As shown in this figure, inside the

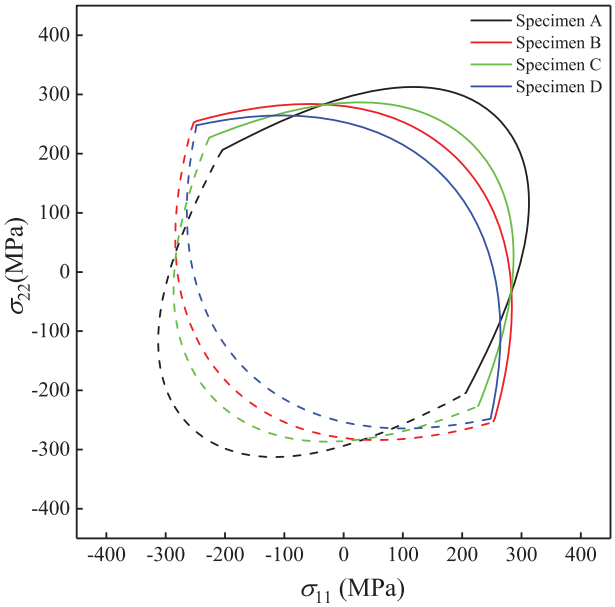

Fig. 3 The strength contours of the four types of specimens. Inside the contour, the specimens were safe. The solid (dash) curves were given by Eq. (30) with the parameters *A*, *C*, and *E* obtained for the uniaxial tensile (compressive) loading.

central closed contour, the specimens were safe. In addition, the central safe areas for the four types of specimens are shown in Fig. 3 for comparison.

5 Concluding Remarks

The conventional strength theories are mainly based on some sort of postulations or empirical consideration. It is also a well-accepted assumption that at the most dangerous point in a specimen, when the maximum stress or some sort of the combination of stresses reaches the critical value, the specimen will fail. In fact, the strength problem is not a local phenomenon [5]; based on the thermodynamics consideration, it is a global property. The failure process is a nonequilibrium evolution process, and the strength criterion of a material can be determined by identifying the critical state of the failure process. However, the classical strength theory can be used very easily in engineering. After the material data are determined using standard specimens with simple loadings, using the classical strength theory, one can predict the strength for any specimens with any sort of loadings. In this work, the aim was to develop an easy-to-use strength theory similar to the classical strength models based on rational thermodynamic consideration [5]. The material constants could be determined with a simple loading test such as uniaxial tensile and compressive loading. Then, based on the rational strength theory, the material strength could be predicted with complex loading. The most valuable point was that the strength theory was not based any sort of empirical considerations such as the classical strength theory. The only assumption was that the dissipated energy density could be expanded into the power form of the plastic deformation when it was small. Therefore, it was a quite general theory. In this study, the predictions were also compared with the experimental data, and the verification gave very positive conclusions. However, the present strength model has some obvious limitations; first, the free energy expression was expressed using the concept of eigenstrains [18,19], which means the damage process could be modeled by the eigenstrains; second, the dissipation energy was expressed by using the Landau–Lifshitz form expansion, which means the eigenstrain, or plastic strain should be infinitesimal, and the material should be isotropic, and with the same tensile and compressive properties.

Finally, the main purpose of this study is to illustrate the fact that the rational nonequilibrium strength theory developed by the author could also be easily used in engineering. Determining the strength of materials has always been a key problem for scientists and engineers. At the present stage, one should formulate strength theories on a rational base, instead of depending on some empirical parameters or criteria. Very successful strength theories in fracture mechanics, such as the energy release rate, the stress intensity factor theory, and the J-integral, have set very good examples. Since the problems of material strength have lasted for several hundred years and the classical theories have been used for a very long time, no one can expect to change the situation overnight. However, such an important problem should not depend on the empirical theory or models. Some new and intrinsic strength characteristics have been found. One should reconsider the strength problem and systematically develop more rational and accurate methodology to evaluate the strength of materials and structures.

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