Meccanica **39:** 63–76, 2004.

Investigation of the Dynamic Behavior of a Griffith Crack in a Piezoelectric Material Strip Subjected to the Harmonic Elastic Anti-plane Shear Waves by Use of the Non-local Theory

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(Received: 8 May 2001; accepted in revised form: 21 January 2003)

Abstract. In this paper, the dynamic behavior of a Griffith crack in a piezoelectric material strip subjected to the harmonic anti-plane shear waves is investigated by use of the non-local theory for impermeable crack surface conditions. To overcome the mathematical difficulties, a one-dimensional non-local kernel is used instead of a two-dimensional one for the anti-plane dynamic problem to obtain the stress and the electric displacement near at the crack tip. By means of the Fourier transform, the problem can be solved with the help of two pairs of dual integral equations. These equations are solved using the Schmidt method. Contrary to the classical solution, it is found that no stress and electric displacement singularity is present near the crack tip. The non-local dynamic elastic solutions yield a finite hoop stress near the crack tip, thus allowing for a fracture criterion based on the maximum dynamic stress hypothesis. The finite hoop stress at the crack tip depends on the crack length, the thickness of the strip, the circular frequency of incident wave and the lattice parameter.

Key words: Harmonic elastic waves, Piezoelectric materials, Non-local theory, Fourier integral transform, Crack, Schmidt method.

1. Introduction

In the theoretical studies of crack problems in piezoelectric materials, several different electric boundary conditions on the crack surfaces have been proposed by numerous researchers [1–12]. For the impermeable crack problems in piezoelectric materials, some significant results have been obtained in [1–6]. The conducting crack problems in the piezoelectric materials were also studied in [7, 8]. Recently, the behavior of two collinear permeable cracks in a piezoelectric layer bonded to two half spaces was studied in [9]. Dunn [10], Zhang and Tong [11] and Sosa and Khutoryansky [12] avoided the common assumption of electric impermeability and utilized more accurate electric boundary conditions at the rim of an elliptical flaw to deal with anti-plane problems in piezoelectricity. They analyzed the effects of electric boundary conditions at the crack surfaces on the fracture mechanics of the piezoelectric materials. However, these solutions contain stress and electric displacement singularities. These phenomena are not reasonable according to the physical nature. In fact, the stress near the crack tip is finite, so beginning with Griffith, all fracture criteria in practice today are based on other considerations, for example, energy, as well as the J-integral.

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In order to remove the stress singularity of classical elastic theory, Eringen [13–15] used the non-local theory to discuss the state of stress near the tip of a sharp line crack in an elastic plane subjected to uniform tension, shear and anti-plane shear loadings. In contrast to this local approach of zero-range internal interactions, the modern non-local continuum mechanics, originated and developed in the last four decades, postulates that the local state at a point is influenced by the action of all particles of the body. In [16], the basic theory of non-local elasticity was stated with emphasis on the difference between the non-local theory and classical continuum mechanics. The basic idea of non-local elasticity is to establish a relationship between macroscopic mechanical quantities and microscopic physical quantities within the framework of continuum mechanics. In [17], the same problem which was treated by Eringen [15] was reworked by use of a somewhat different approach. The dynamic behaviors of a Griffith crack and two cracks in the elastic materials subjected to the harmonic anti-plane shear waves were investigated by use of the non-local theory in [18, 19]. The scattering behaviors of the harmonic stress waves by the Mode-I and Mode-II cracks were studied by use of the non-local theory in [20, 21]. These solutions did not contain any stress singularity, thus resolving a fundamental problem that has remained unsolved for over many years. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way. To our knowledge, the dynamic electro-elastic behavior of the piezoelectric material strip with a crack subjected to harmonic elastic anti-plane shear wave and in-plane electric loading has not been studied by use of the non-local theory.

In the present paper, the scattering of the harmonic anti-plane shear elastic waves by a Griffith impermeable crack in piezoelectric material strip is investigated by use of the non-local theory. The traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects. To obtain the theoretical solution, and discussing the probability of using the non-local theory to solve the dynamic fracture problem in the piezoelectric material strip, one has to accept some assumptions as in [22, 23], that is, a one-dimensional non-local kernel function was used instead of a two-dimensional kernel function for the anti-plane dynamic problem. Obviously, the assumption should be further investigated to satisfy the realistic condition. Fourier transform is applied and a mixed boundary value problem is reduced to two pairs of dual integral equations. To solve the dual integral equations, the crack surface displacement and electric potential are expanded in a series of Jacobi polynomials by means of the Schmidt method [24]. This process is quite different from that adopted in previous works as mentioned above [1–15]. As expected, the solution in this paper does not contain the stress and electric displacement singularities near the crack tip, thus clearly indicating the physical nature of the problem.

2. Basic Equations of the Non-local Piezoelectric Materials

For the anti-plane shear problem, the basic equations of linear, homogeneous, isotropic, non-local piezoelectric materials, with vanishing body force are [15, 16, 19, 25]

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2}, \tag{1}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0, \tag{2}$$

$$\tau_{kz}(X,t) = \int_V [c'_{44}(|X'-X|)w_{,k}(X',t) + e'_{15}(|X'-X|)\phi_{,k}(X',t)]\,\mathrm{d}V(X') \quad (k=x,y), \tag{3}$$

$$D_k(X,t) = \int_V [e'_{15}(|X'-X|)w_{,k}(X',t) - \varepsilon'_{11}(|X'-X|)\phi_{,k}(X',t)]\,\mathrm{d}V(X') \quad (k=x,y), \tag{4}$$

where the only difference with classical elastic theory and the piezoelectric theory is in the stress and the electric displacement constitutive equations (3) and (4) in which the stress $\tau_{zk}(X,t)$ and the electric displacement $D_k(X,t)$ at a point X depends on $w_{,k}(X,t)$ and $\phi_{,k}(X,t)$, at all points of the body. w and ϕ are the mechanical displacement and electric potential. For homogeneous and isotropic piezoelectric materials there exist only three material parameters, $c'_{44}(|X'-X|)$, $e'_{15}(|X'-X|)$ and $\varepsilon'_{11}(|X'-X|)$ which are functions of the distance $|X'-X|$. ρ is the density of the piezoelectric materials. The integrals in (3) and (4) are over the volume V of the body enclosed within a surface ∂V.

As discussed in [26, 27], it can be assumed in the form of $c'_{44}(|X'-X|)$, $e'_{15}(|X'-X|)$ and $\varepsilon'_{11}(|X'-X|)$ for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves the following has been found to be very useful

$$(c'_{44}, e'_{15}, \varepsilon'_{11}) = (c_{44}, e_{15}, \varepsilon_{11})\alpha(|X'-X|), \tag{5}$$

$\alpha(|X'-X|)$ is known as the influence function, and is the functions of the distance $|X'-X|$. c_{44}, e_{15} and ε_{11} are the shear modulus, piezoelectric coefficient and dielectric parameter, respectively.

Substitution of equation (5) into equations (3) and (4) yields

$$\tau_{kz}(X,t) = \int_V \alpha(|X'-X|)\sigma_{kz}(X',t)\,\mathrm{d}V(X') \quad (k=x,y), \tag{6}$$

$$D_k(X,t) = \int_V \alpha(|X'-X|)D_k^c(X',t)\,\mathrm{d}V(X') \quad (k=x,y), \tag{7}$$

where

$$\sigma_{kz} = c_{44}w_{,k} + e_{15}\phi_{,k}, \tag{8}$$

$$D_k^c = e_{15}w_{,k} - \varepsilon_{11}\phi_{,k}. \tag{9}$$

The expressions (8) and (9) are the classical piezoelectric material constitutive equations.

3. The Crack Model and the Solution

In the present paper, the problem of an infinite long piezoelectric material strip with width $2h$, containing a crack parallel to the edges of the strip is considered. The crack occupies the region $-l \leqslant x \leqslant l$, $y=0$, and the geometry of the problem is shown in Figure 1. As discussed in [25, 28], when the harmonic wave is vertically incident to the crack, the magnitudes of the

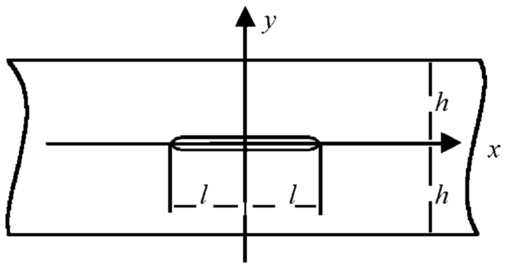

Figure 1. Crack in a piezoelectric material strip.

stress and the electric displacement inside the crack should be constants. Let ω be the circular frequency of the incident wave. $-\tau_0$ is a magnitude of the incident wave. In what follows, the time dependence term $e^{-i\omega t}$ of all variables will be suppressed as commonly used technology as discussed in [25, 28]. The magnitudes of all variables are only considered in what follows. It is further supposed that the two faces of the crack do not come in contact during vibrations. The piezoelectric boundary-value problem for anti-plane shear is considerably simplified if we only consider the out-of-plane displacement and the in-plane electric fields. When the harmonic anti-plane shear wave is vertically incident to the crack, the boundary conditions on the crack surfaces at $y = 0$ are (Here, we just consider the perturbation field.)

$$\tau_{yz}(x, 0) = -\tau_0, \qquad D_y(x, 0) = -D_0, \quad |x| \leqslant l, \tag{10}$$

$$\tau_{yz}(x, \pm h) = D_y(x, \pm h) = 0, \quad |x| \leqslant \infty, \tag{11}$$

$$w(x, 0) = \phi(x, 0) = 0, \quad |x| > l, \tag{12}$$

$$w(x, y) = \phi(x, y) = 0 \quad \text{for } (x^2 + y^2)^{1/2} \to \infty, \tag{13}$$

where τ_0 and D_0 are positive. Substituting equations (6) and (7) into equations (1) and (2), respectively, using Green–Gauss theorem, it can be obtained [15]:

$$\iint_V \alpha(|x' - x|, |y' - y|)[c_{44}\nabla^2 w(x', y') + e_{15}\nabla^2\phi(x', y')]\,dx'\,dy' \\ - \int_{-l}^{l} \alpha(|x' - x|, 0)[\sigma_{yz}(x', 0)]\,dx' = -\rho\omega^2 w(x, y), \tag{14}$$

$$\iint_V \alpha(|x' - x|, |y' - y|)[e_{15}\nabla^2 w(x', y') - \varepsilon_{11}\nabla^2\phi(x', y')]\,dx'\,dy' \\ - \int_{-l}^{l} \alpha(|x' - x|, 0)[D_y^c(x', 0)]\,dx' = 0, \tag{15}$$

where the boldface bracket indicates a jump at the crack line, that is,

$$\begin{aligned} [\sigma_{yz}(x', 0)] &= \sigma_{yz}(x', 0^+) - \sigma_{yz}(x', 0^-), \\ [D_y^c(x', 0)] &= D_y^c(x', 0^+) - D_y^c(x', 0^-). \end{aligned}$$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \leqslant x \leqslant \infty$,

$0 \leqslant y \leqslant h$ only. Under the applied anti-plane shear load on the unopened surfaces of the crack, the displacement field and the electric displacement possess the following symmetry regulations

$$w(x, -y) = -w(x, y), \quad \phi(x, -y) = -\phi(x, y). \tag{16}$$

Using the equation (16), we find that

$$[\sigma_{yz}(x, 0)] = 0, \tag{17}$$

$$[D_y^c(x, 0)] = 0. \tag{18}$$

Hence the line integrals in (14) and (15) vanish. By taking the Fourier transform of (14) and (15) with respect to x', it can be shown that

$$\begin{aligned}\int_0^\infty \bar{\alpha}(|s|, |y' - y|)\bigg\{ c_{44}\bigg[\frac{\mathrm{d}^2\bar{w}(s, y')}{\mathrm{d}y^2} - s^2\bar{w}(s, y')\bigg] \\ + e_{15}\bigg[\frac{\mathrm{d}^2\bar{\phi}(s, y')}{\mathrm{d}y^2} - s^2\bar{\phi}(s, y')\bigg]\bigg\}\,\mathrm{d}y' = -\rho\omega^2\bar{w},\end{aligned} \tag{19}$$

$$\begin{aligned}\int_0^\infty \bar{\alpha}(|s|, |y' - y|)\bigg\{ e_{15}\bigg[\frac{\mathrm{d}^2\bar{w}(s, y')}{\mathrm{d}y^2} - s^2\bar{w}(s, y')\bigg] \\ - \varepsilon_{11}\bigg[\frac{\mathrm{d}^2\bar{\phi}(s, y')}{\mathrm{d}y^2} - s^2\bar{\phi}(s, y')\bigg]\bigg\}\,\mathrm{d}y' = 0.\end{aligned} \tag{20}$$

Here a superposed bar indicates the Fourier transform through the present paper. For the even function, the Fourier transform can be written as follow form, that is,

$$\bar{f}(s) = \int_0^\infty f(x)\ \cos(sx)\,\mathrm{d}x, \quad f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}(s)\ \cos(sx)\,\mathrm{d}s.$$

For the odd function, the Fourier transform can be written as in the following form, that is,

$$\bar{f}(s) = \int_0^\infty f(x)\ \sin(sx)\,\mathrm{d}x, \quad f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}(s)\ \sin(sx)\,\mathrm{d}s.$$

What now remains is to solve the integrodifferential equations (19) and (20) for the function w and ϕ.

It is impossible to obtain a rigorous solution at the present stage for equations (19) and (20). It seems obviously that in the solution of such a problem we encounter seriously if not unsurmountable mathematical difficulties and will have to resort to an approximate procedure. In the given problem, according to the assumptions as in [22, 23], the non-local interaction in y direction is ignored. So the non-local influence function $\bar{\alpha}(s, y)$ can be separated as follows:

$$\bar{\alpha}(|s|, |y' - y|) = \bar{\alpha}_0(s)\delta(y' - y). \tag{21}$$

The non-local function α will depend on a characteristic length ratio a/l, where a is an internal characteristic length (e.g., lattice parameter, granular distance. In this paper, a represents the lattice parameter) and l is an external characteristic length (e.g., crack length, wave-length. In this paper, l represents the crack length). By matching the dispersion curves of plane waves with those of atomic lattice dynamics (or experiments), we can determine the non-local modulus function α for given material. As discussed in [13–15, 22, 23, 29], the form of $\alpha_0(s)$ in (21) can be written as follows:

$$\alpha_0 = \chi_0 \exp\left(-\left(\frac{\beta}{a}\right)^2 (x'-x)^2\right), \quad \chi_0 = \frac{\beta}{a\sqrt{\pi}}, \tag{22}$$

where β is a constant (here β is a constant appropriate to each material) and a is the lattice parameter. So it can be obtained

$$\bar{\alpha}_0(s) = \exp\left(-\frac{(sa)^2}{(2\beta)^2}\right), \tag{23}$$

and $\bar{\alpha}_0(s) = 1$ for the limit $a \to 0$, so that Equations (19) and (20) reduces to the well-known equation of the classical theory.

From equations (19) and (20) and use of the equation (21), we have

$$\bar{\alpha}_0(s)\left\{c_{44}\left[\frac{\mathrm{d}^2\bar{w}(s,y)}{\mathrm{d}y^2} - s^2\bar{w}(s,y)\right] + e_{15}\left[\frac{\mathrm{d}^2\bar{\phi}(s,y)}{\mathrm{d}y^2} - s^2\bar{\phi}(s,y)\right]\right\} = -\rho\omega^2\bar{w}, \tag{24}$$

$$e_{15}\left[\frac{\mathrm{d}^2\bar{w}(s,y)}{\mathrm{d}y^2} - s^2\bar{w}(s,y)\right] - \varepsilon_{11}\left[\frac{\mathrm{d}^2\bar{\phi}(s,y)}{\mathrm{d}y^2} - s^2\bar{\phi}(s,y)\right] = 0. \tag{25}$$

The solution of equations (24) and (25) does not present difficulties, it can be written as follows, respectively. ($y \geqslant 0$):

$$\begin{aligned} w(x,y) &= \frac{2}{\pi}\int_0^\infty [A_1(s)\,\mathrm{e}^{-\gamma y} + A_2(s)\,\mathrm{e}^{\gamma y}]\cos(xs)\,\mathrm{d}s, \\ \phi(x,y) &= \frac{e_{15}}{\varepsilon_{11}} w(x,y) + \frac{2}{\pi}\int_0^\infty [B_1(s)\,\mathrm{e}^{-sy} + B_2(s)\,\mathrm{e}^{sy}]\cos(xs)\,\mathrm{d}s, \end{aligned} \tag{26}$$

where $\gamma^2 = s^2 - \omega^2/c^2\bar{\alpha}_0(s)$, $c^2 = \mu/\rho$, $\mu = c_{44} + \frac{e_{15}^2}{\varepsilon_{11}}$. $A_1(s)$, $A_2(s)$, $B_1(s)$ and $B_2(s)$ are to be determined from the boundary conditions.

Because of the symmetry, it suffices to consider the problem in the first quadrant only. According to the boundary conditions (10–12), it can be obtained

$$\frac{2}{\pi}\int_0^\infty \bar{\alpha}_0(s)\gamma\frac{1-\exp(-2\gamma h)}{1+\exp(-2\gamma h)}A(s)\cos(sx)\,\mathrm{d}s = \frac{1}{\mu}\left(\tau_0 + \frac{e_{15}D_0}{\varepsilon_{11}}\right), \quad |x| \leqslant l, \tag{27}$$

$$\frac{2}{\pi}\int_0^\infty A(s)\cos(sx)\,\mathrm{d}s = 0, \quad |x| > l, \tag{28}$$

and

$$\frac{2}{\pi}\int_0^\infty \bar{\alpha}_0(s)s\frac{1-\exp(-2sh)}{1+\exp(-2sh)}B(s)\cos(sx)\,\mathrm{d}s = -\frac{D_0}{\varepsilon_{11}}, \quad |x| \leqslant l, \tag{29}$$

$$\frac{2}{\pi}\int_0^\infty B(s)\cos(sx)\,\mathrm{d}s = 0, \quad |x| > l. \tag{30}$$

The relationships between the functions $A(s)$, $B(s)$, $A_1(s)$, $A_2(s)$, $B_1(s)$ and $B_2(s)$ are obtained by applying a Fourier sine transform [30] to equation (11):

$$\begin{aligned} A(s) &= [1 + \mathrm{e}^{-2\gamma h}]A_1(s), \qquad A_2(s) = \mathrm{e}^{-2\gamma h} A_1(s), \\ B(s) &= [1 + \mathrm{e}^{-2sh}]B_1(s), \qquad B_2(s) = \mathrm{e}^{-2sh} B_1(s). \end{aligned}$$

To determine the unknown functions $A(s)$ and $B(s)$, the above two pairs of dual-integral equations (27)–(30) must be solved. It can seen that the integrands of the dual integral equations both in [15] and in the present paper have the same behavior for $s \to \infty$, that is, they do not tend to a constant for $s \to \infty$. The dual integral equations (27)–(30) cannot be transformed into a Fredholm integral equation of the second kind, because the kernel of the Fredholm integral equation of the second kind in [15] is divergent. It can be rewritten as following:

$$h(x) + \int_0^1 h(u)L(x,u)\,\mathrm{d}u = g(x), \tag{31}$$

where $g(x)$ is known function, $h(x)$ is unknown function.

The kernel of the above Fredholm integral equation of the second kind can be written as follows:

$$L(x,u) = (xu)^{1/2} \int_0^\infty tk(\varepsilon t) J_0(xt) J_0(ut)\,\mathrm{d}t, \quad 0 \leqslant x, \quad u \leqslant 1, \tag{32}$$

where $J_n(x)$ is the Bessel function of order n.

$$k(\varepsilon t) = -\Phi(\varepsilon t), \lim_{t\to\infty} k(\varepsilon t) \neq 0 \quad \text{for } \varepsilon = \frac{a}{2\beta l} \neq 0, \tag{33}$$

where l is the length of the crack,

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{4}\pi\right) \quad \text{for } x \gg 0. \tag{34}$$

The limit of $tk(\varepsilon t)J_0(xt)J_0(ut)$ is not equal to zero for $t \to \infty$. So the kernel $L(x,u)$ in [15] is divergent. Of course, the dual integral equations (27)–(30) can be considered to be a single integral equation of the first kind with a discontinuous kernel [13]. It is well known in the literature that integral equations of the first kind are generally ill-posed in the sense of Hadamard, for example, small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. In this paper, the Schmidt method [24] was used to overcome the difficulty. Here the Schmidt method can be used to solve the dual integral equations (27)–(30). The displacement w and the electric potential ϕ are represented by the following series:

$$w(x,0) = \sum_{n=1}^\infty a_n P_{2n-2}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{1/2}, \quad \text{for } -l \leqslant x \leqslant l,\ y = 0, \tag{35}$$

$$w(x,0) = 0, \quad \text{for } |x| > l,\ y = 0, \tag{36}$$

$$\phi(x,0) = \sum_{n=1}^\infty b_n P_{2n-2}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{1/2}, \quad \text{for } -l \leqslant x \leqslant l,\ y = 0, \tag{37}$$

$$\phi(x, 0) = 0, \quad \text{for } |x| > l,\ y = 0, \tag{38}$$

where a_n and b_n are unknown coefficients to be determined and $P_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial [30]. The Fourier transform of equations (35)–(38) are [31]

$$A(s) = \bar{w}(s, 0) = \sum_{n=1}^{\infty} a_n G_n \frac{1}{s} J_{2n-1}(sl), \tag{39}$$

$$B(s) = \bar{\phi}(s, 0) - \frac{e_{15}}{\varepsilon_{11}} \bar{w}(s, 0) = \sum_{n=1}^{\infty} \left(b_n - \frac{e_{15}}{\varepsilon_{11}} a_n \right) G_n \frac{1}{s} J_{2n-1}(sl), \tag{40}$$

$$G_n = 2\sqrt{\pi}(-1)^{n-1} \frac{\Gamma(2n - 1/2)}{(2n - 2)!}, \tag{41}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting equations (39) and (40) into equations (27)–(30), respectively, equations (28) and (30) can be automatically satisfied, respectively. Then equations (27) and (29) reduce to the form, respectively.

$$\sum_{n=1}^{\infty} a_n G_n \int_0^{\infty} \bar{\alpha}_0(s) \frac{\gamma[1 - e^{-2\gamma h}]}{s[1 + e^{-2\gamma h}]} J_{2n-1}(sl) \cos(sx)\, ds = \frac{\pi}{2\mu} \tau_0 (1 + \lambda), \tag{42}$$

$$\sum_{n=1}^{\infty} \left(b_n - \frac{e_{15}}{\varepsilon_{11}} a_n \right) G_n \int_0^{\infty} \bar{\alpha}_0(s) \frac{1 - e^{-2sh}}{1 + e^{-2sh}} J_{2n-1}(sl) \cos(sx)\, ds = -\frac{\pi D_0}{2\varepsilon_{11}}, \tag{43}$$

where $\lambda = e_{15} D_0 / \varepsilon_{11} \tau_0$.

The semi-infinite integral in equation (42) can be evaluated numerically by Filon's method [32], except for singularities in the integrands of the integrals in equation (42). These singularities are poles that occur in the complex plane at the zero of the function $1 + \exp(-2\gamma h)$, such as $2\gamma h = i\pi, 3i\pi, 5i\pi, \ldots$. All poles depend on the material, the incident wave frequency ω and the lattice parameter. It may be noted that the integral of equation (42) is not convergent at these poles. However, there is no pole for $\omega/c < \pi/2h$. So the integral of equation (42) is convergent at these poles for $\omega/c < \pi/2h$. In this paper, it is only discussed the case of $\omega/c < \pi/2h$. From the reference [28], this case is consistent with the statement that the only shear waves with $\omega/c < \pi/2h$ can be propagated in an elastic strip of width $2h$. This fact is in agreement with the well-known results of frequencies less than a parameter depending on the width of the strip can propagate. For $\omega/c > \pi/2h$, it should be further investigated. For large s, the integrands of equations (42) and (43) almost decrease exponentially and can be evaluated numerically by Filon's method [32]. Equations (42) and (43) can now be solved for the coefficients a_n and b_n by the Schmidt's method [24] for $\omega/c < \pi/2h$. For brevity, the equation (42) can be rewritten as (the equation (43) can be solved using a similar method as following):

$$\sum_{n=1}^{\infty} a_n E_n(x) = U(x), \quad -l < x < l, \tag{44}$$

where $E_n(x)$ and $U(x)$ are known functions and the coefficients a_n are to be determined. A set of functions $P_n(x)$ which satisfy the orthogonality condition

$$\int_{-l}^{l} P_m(x)P_n(x)\,\mathrm{d}x = N_n\delta_{mn}, \quad N_n = \int_{-l}^{l} P_n^2(x)\,\mathrm{d}x \tag{45}$$

can be constructed from the function, $E_n(x)$, such that

$$P_n(x) = \sum_{i=1}^{n} \frac{M_{in}}{M_{nn}} E_i(x), \tag{46}$$

where M_{ij} is the cofactor of the element d_{ij} of D_n, which is defined as

$$D_n = \begin{bmatrix} d_{11}, d_{12}, d_{13}, \ldots, d_{1n} \\ d_{21}, d_{22}, d_{23}, \ldots, d_{2n} \\ d_{31}, d_{32}, d_{33}, \ldots, d_{3n} \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ d_{n1}, d_{n2}, d_{n3}, \ldots, d_{nn} \end{bmatrix}, \quad d_{ij} = \int_{-l}^{l} E_i(x)E_j(x)\,\mathrm{d}x. \tag{47}$$

Using equations (44)–(47), we obtain

$$a_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \quad \text{with} \quad q_j = \frac{1}{N_j}\int_{-l}^{l} U(x)P_j(x)\,\mathrm{d}x. \tag{48}$$

4. Numerical Calculations and Discussion

From the works [4, 9, 20, 21, 33, 34], it can be seen that the Schmidt method is performed satisfactorily if the first 10 terms of the infinite series [44] are retained. The behavior of the maximum dynamic stress stays steady with the increasing number in terms in [44]. Although we can determine the entire dynamic the stress field and the electric displacement from the coefficients a_n and b_n, it is important in fracture mechanics to determine the dynamic stress τ_{yz} and the electric displacement D_y in the vicinity of the crack tip. τ_{yz} and D_y along the crack line can be expressed, respectively, as

$$\begin{aligned} \tau_{yz}(x,0) \;=\; -\frac{2}{\pi}\sum_{n=1}^{\infty}\bigg[&\mu a_n G_n \int_0^{\infty} \bar{\alpha}_0(s)\frac{\gamma[1-\mathrm{e}^{-2\gamma h}]}{s[1+\mathrm{e}^{-2\gamma h}]} J_{2n-1}(sl)\,\cos(xs)\,\mathrm{d}s \\ &+ e_{15}\left(b_n - \frac{e_{15}}{\varepsilon_{11}}a_n\right)G_n \int_0^{\infty} \bar{\alpha}_0(s)\frac{1-\mathrm{e}^{-2sh}}{1+\mathrm{e}^{-2sh}} J_{2n-1}(sl)\,\cos(xs)\,\mathrm{d}s\bigg], \end{aligned} \tag{49}$$

$$D_y(x,0) = -\frac{2}{\pi}\sum_{n=1}^{\infty}(e_{15}a_n - \varepsilon_{11}b_n)G_n \int_0^{\infty} \bar{\alpha}_0(s)\frac{1-\mathrm{e}^{-2sh}}{1+\mathrm{e}^{-2sh}} J_{2n-1}(sl)\,\cos(xs)\,\mathrm{d}s. \tag{50}$$

For $a = 0$ at $x = l$, we have the classical stress and the electric displacement singularities. However, so long as $a \neq 0$, the semi-infinite integration and the series in the equations (49) and (50) are convergent for any variable x. The equations (49) and (50) give a finite stress and a finite electric displacement all along $y = 0$, so there are no stress and electric displacement singularities at the crack tip. At $-l < x < l$, τ_{yz}/τ_0 is very close to unity, and for $x > l$, τ_{yz}/τ_0 and D_y/D_0 possess finite values diminishing from a finite value at $x = l$ to zero at $x = \infty$. Since $a/2\beta l > 1/100$ represents a crack length of less than 100 atomic distances as stated in [15], and such submicroscopic sizes other serious questions arise regarding the interatomic arrangements and force laws, we do not pursue solutions valid at such small crack sizes. The semi-infinite numerical integrals, which occur, are evaluated easily by Filon's method and Simpson methods because the rapid diminution of the integrands. From the equations (42), (43), (49) and (50), it can be found that the dimensionless stress is independent of the material parameters. However, the electric field is found to be independent of the circular frequency of the incident wave and the wave velocity. It depends on the material constants of the piezoelectric materials. The piezoelectric layer is assumed to be the commercially available piezoelectric PZT-4 or PZT-5H. The material constants of PZT-4 are $c_{44} = 2.56 \times 10^{10}\,\mathrm{N/m^2}$, $e_{15} = 12.7\,\mathrm{C/m^2}$, $\varepsilon_{11} = 64.6 \times 10^{-10}\,\mathrm{C/Vm^2}$, $\rho = 7500\,\mathrm{kg/m^3}$, respectively. The material constants of PZT-5H are $c_{44} = 2.3 \times 10^{10}\,\mathrm{N/m^2}$, $e_{15} = 17.0\,\mathrm{C/m^2}$, $\varepsilon_{11} = 150.4 \times 10^{-10}\,\mathrm{C/Vm^2}$, $\rho = 7500\,\mathrm{kg/m^3}$, respectively.

The results are plotted in Figures 2–9. The following observations are very significant:

(i) the traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects.
(ii) the maximum stress and the electric displacement do not occur at the crack tip, but slightly away from it. This phenomenon has been thoroughly substantiated by Eringen [35]. The

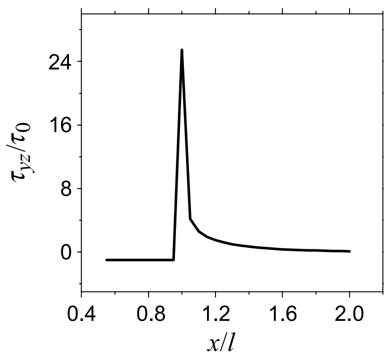

Figure 2. The stress along crack line *versus* x/l for $\omega l/c = 0.5$, $a/2\beta = 0.001$, $\lambda = 0.4$, $h = 0.5$, $l = 1.0$.

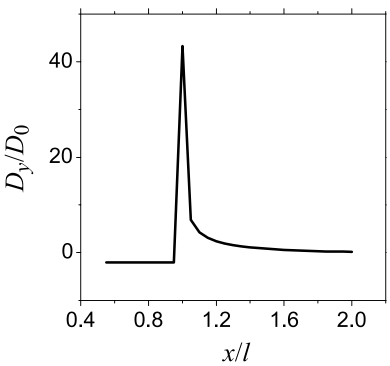

Figure 3. The electric displacement along crack line *versus* x/l for $\omega l/c = 0.5$, $a/2\beta = 0.001$, $\lambda = 0.4$, $h = 0.5$, $l = 1.0$ (PZT-4).

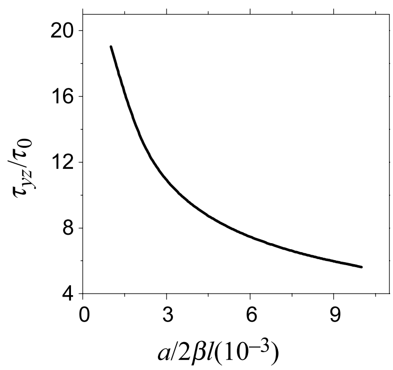

Figure 4. The stress at crack tip *versus* $a/2\beta$ for $\omega l/c = 0.3$, $h = 1.0$, $\lambda = 0.4$.

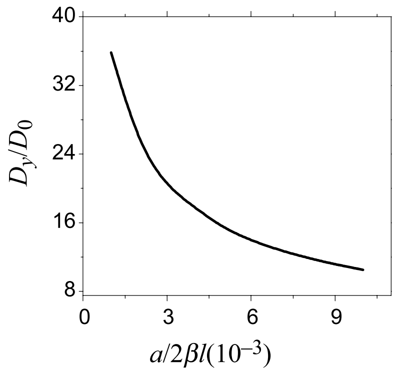

Figure 5. The electric displacement at crack tip *versus* $a/2\beta$ for $h = 1.0$, $\omega l/c = 0.3$, $\lambda = 0.4$ (PZT-4).

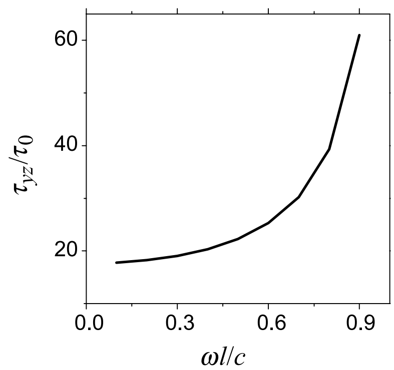

Figure 6. The stress at crack tip *versus* ω for $h = 1.0$, $a/2\beta = 0.001$, $\lambda = 0.4$.

maximum stress and the electric displacement are finite. The distance between the crack tip and the maximum stress point is very small, and it depends on the crack length and the lattice parameter. Contrary to the classical piezoelectric theory solution, it is found that no stress and electric displacement singularities are present at the crack tip, and also the present results converge to the classical ones when far away from the crack tip as shown in Figures 2 and 3. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way.

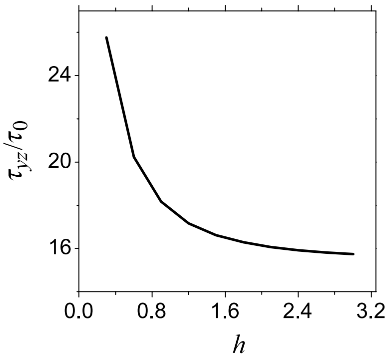

Figure 7. The stress at crack tip *versus* h for $\omega l/c = 0.1$, $a/2\beta = 0.001$, $\lambda = 0.4$, $l = 1.0$.

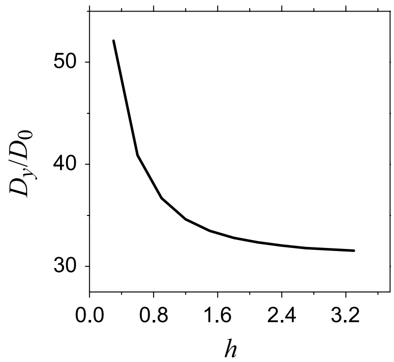

Figure 8. The electric displacement at crack tip *versus* h for $\omega l/c = 0.1$, $a/2\beta = 0.001$, $\lambda = 0.4$, $l = 1.0$ (PZT-4).

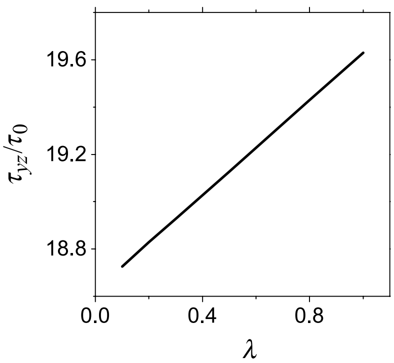

Figure 9. The stress at crack tip *versus* λ for $\omega l/c = 0.3$, $a/2\beta = 0.001$, $h = 1.0$, $l = 1.0$.

(iii) from equations (49) and (50), it can be obtained that the dynamic stress and the electric displacement at the crack tip become infinite as the lattice parameter $a \to 0$. This is the classical continuum limit of square root singularity. For the local theory, it can only obtain the stress and electric displacement intensity factors for the variation with $\omega l/c$.

(iv) from the results as shown in Figures 4 and 5, it can be found that the value of the stress and the electric displacement concentrations (at the crack tip) increase with increasing of the crack length. Noting this fact, experiments indicate that the piezoelectric materials with smaller cracks are more resistant to fracture than those with larger cracks (see [15]). In addition, the value of the stress and the electric displacement concentrations (at the crack tip) decrease with increasing of the lattice parameter.

(v) from the equations (42), (43), (49) and (50), it can be found that the dimensionless stress is independent of the material parameters. It just depends on the length of the crack, the lattice parameter, the thickness of the strip, the circular frequency of the incident wave, the electric loading and the wave velocity. However, the electric field is found to be independent of the circular frequency of the incident wave and the wave velocity. It just depends on the length of the crack, the thickness of the strip, the electric loading, the material constants of the piezoelectric materials and the lattice parameter.

(vi) the dynamic stress at the crack tip tends to increase with increasing of the frequency for $\omega/c < 0.9$ as shown in Figure 6 (In this paper, it is only discussed the case of $\omega/c < \pi/2h$.).

(vii) from the results as shown in Figures 7 and 8, it can be found that the dynamic stress and the dynamic electric displacement at the crack tip tend to decrease with increasing of the thickness of the strip. For $h \geqslant 2.5$, the decreasing tendency becomes slowly, that is, the influence of the thickness of the piezoelectric strip to the results becomes smaller.

(viii) from the results as shown in Figure 9, it can be obtained that the dynamic stress at the crack tip tends to increase almost linearly with increasing of the electric loading.

Acknowledgements

The authors are grateful for financial support from the Natural Science Foundation of Hei Long Jiang Province, the National Natural Science Foundation of China through the Key Program (50232030), the National Natural Science Foundation of China (10172030), the SRF for ROCS, SEM and the Multidiscipline Scientific Research Foundation of Harbin Institute of Technology (HIT.MD.2001.39).

References

1. Deeg, W.E.F., '*The Analysis of Dislocation, Crack and Inclusion Problems in Piezoelectric Solids*', PhD Thesis, Stanford University, 1980.
2. Pak, Y.E., 'Crack extension force in a piezoelectric material', *ASME J. Appl. Mech.* **57** (1990) 647–653.
3. Sosa, H.A. and Pak, Y.E., 'Three-dimensional eigenfunction analysis of a crack in a piezoelectric ceramics', *Int. J. Solids Struct.* **26** (1990) 1–15.
4. Zhou, Z.G., Chen, J.Y. and Wang, B., 'Analysis of two collinear cracks in a piezoelectric layer bonded to two half spaces subjected to anti-plane shear', *Meccanica* **35** (2000) 443–456.
5. Suo, Z., Kuo, C.-M., Barnett, D.M. and Willis, J.R., 'Fracture mechanics for piezoelectric ceramics', *J. Mech. Phys. Solids* **40** (1992) 739–765.
6. Gao, H., Zhang, T.Y. and Tong, P., 'Local and global energy rates for an elastically yielded crack in piezoelectric ceramics', *J. Mech. Phys. Solids* **45** (1997) 491–510.
7. McMeeking, R.M., 'On mechanical stress at cracks in dielectrics with application to dielectric breakdown', *J. Appl. Phys.* **62** (1989) 3316–3122.
8. Suo, Z., 'Models for breakdown-resistant dielectric and ferroelectric ceramics', *J. Mech. Phys. Solids* **41** (1993) 1155–1176.
9. Zhou, Z.G., Liang, J. and Wang, B., 'Two collinear permeable cracks in a piezoelectric layer bonded to two half spaces', *Meccanica* (2002) (in press).
10. Dunn, M.L., 'The effects of crack face boundary conditions on the fracture mechanics of piezoelectric solids', *Eng. Fract. Mech.* **48** (1994) 25–39.
11. Zhang, T.Y. and Tong, P., 'Fracture mechanics for a mode III crack in a piezoelectric material', *Int. J. Solids Struct.* **33** (1996) 343–359.

12. Sosa, H. and Khutoryansky, N., 'Transient dynamic response of piezoelectric bodies subjected to internal electric impulses', *Int. J. Solids Struct.* **36** (1999) 5467–5484.
13. Eringen, A.C., Speziale, C.G. and Kim, B.S., 'Crack tip problem in non-local elasticity', *J. Mech. Phys. Solids* **25** (1977) 339–355.
14. Eringen, A.C., 'Linear crack subject to shear', *Int. J. Fract*. **14** (1978) 367–379.
15. Eringen, A.C., 'Linear crack subject to anti-plane shear', *Eng. Fract. Mech.* **12** (1979) 211–219.
16. Pan, K.L. and Takeda, N., 'Non-local stress field of interface dislocations', *Arch. Appl. Mech.* **68** (1998) 179–184.
17. Zhou, Z.G., Han, J.C. and Du, S.Y., 'Investigation of a Griffith crack subject to anti-plane shear by using the non-local theory', *Int. J. Solids Struct.* **36** (1999) 3891–3901.
18. Zhou, Z.G., Wang, B. and Du, S.Y., 'Scattering of harmonic anti-plane shear waves by a finite crack by using the non-local theory', *Int. J. Fract.* **91** (1998) 13–22.
19. Zhou, Z.G. and Jia, D.B., 'The scattering of harmonic elastic anti-plane waves by two collinear symmetric cracks in infinite long strip using the non-local theory', *Mech. Res. Comm.* **27** (2000) 307–318.
20. Zhou, Z.G., Han, J.C. and Du, S.Y., 'Investigation of the scattering of harmonic elastic waves by a finite crack using the non-local theory', *Mech. Res. Comm.* **25** (1998) 519–528.
21. Zhou, Z.G. and Shen, Y.P., 'Investigation of the scattering of harmonic shear waves by two collinear cracks using the non-local theory', *Acta Mech.* **135** (1999) 169–179.
22. Nowinski, J.L., 'On non-local aspects of the propagation of love waves', *Int. J. Eng. Sci.* **22** (1984) 383–392.
23. Nowinski, J.L., 'On non-local theory of wave propagation in elastic plates', *ASME J. Appl. Mech.* **51** (1984), 608–613.
24. Morse, P.M. and Feshbach, H., *Methods of Theoretical Physics*, Vol. 1, McGraw-Hill, New York, 1958, pp. 925–937.
25. Narita, K. and Shindo, Y., 'Scattering of anti-plane shear waves by a finite crack in piezoelectric laminates', *Acta Mech.* **134** (1999) 27–43.
26. Eringen, A.C. 'Non-local elasticity and waves', in: Thoft-Christensen, P. (ed.), *Continuum Mechanics Aspects of Geodynamics and Rock Fracture Mechanics*, Dordrecht, Holland, 1974, pp. 81–105.
27. Eringen, A.C., 'Continuum mechanics at the atomic scale', *Crys. Latt. Def.* **7** (1977) 109–130.
28. Srivastava, K.N., Palaiya, R.M. and Karaulia, D.S., 'Interaction of shear waves with two coplanar Griffith cracks situated in an infinitely long elastic strip', *Int. J. Fract.* **23** (1983) 3–14.
29. Eringen, A.C., 'On differential equations of non-local elasticity and solutions of screw dislocation and surface waves', *J. Appl. Phys.* **54** (1983) 4703–4711.
30. Gradshteyn, I.S. and Ryzhik, I.M., *Table of Integral, Series and Products*, Academic Press, New York, 1980, pp. 1025–1031.
31. Erdelyi, A. (ed.), *Tables of Integral Transforms*, Vol. 1, McGraw-Hill, New York, 1954, pp. 101–128.
32. Amemiya, A. and Taguchi, T., *Numerical Analysis and Fortran*, Maruzen, Tokyo, 1969, pp. 346–372.
33. Itou, S., 'Three-dimensional waves propagation in a cracked elastic solid', *ASME J. Appl. Mech.* **45** (1978) 807–811.
34. Itou, S., 'Three-dimensional problem of a running crack', *Int. J. Eng. Sci.* **17** (1979) 59–71.
35. Eringen, A.C., 'Interaction of a dislocation with a crack', *J. Appl. Phys.* **54** (1983) 6811–6819.