On the Dynamic Growth of a Spherical Inclusion With Dilatational Transformation Strain in Infinite Elastic Domain

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In this paper, the dynamic effect was incorporated into the initiation and propagation process of a transformation inclusion. Based on the time-varying propagation equation of a spherical transformation inclusion with pure dilatational eigenstrain, the dynamic elastic fields both inside and outside the inclusion were derived explicitly, and it is found that when the transformation region expands at a constant speed, the strain field inside the inclusion is time-independent and uniform for uniform eigenstrain. Following the basic ideas of crack propagation problems in dynamic fracture mechanics, the reduction rate of the mechanical part of the free energy accompanying the growth of the transformation inclusion was derived as the driving force for the move of the interface. Then the equation to determine the propagation speed was established. It is found that there exists a steady speed for the growth of the transformation inclusion when time is approaching infinity. Finally the relation between the steady speed and the applied hydrostatic stress was derived explicitly.

Introduction

Martensitic-type phase transformations can happen in many kinds of solid materials. These materials can be certain metals, such as shape memory alloys and TRIP steels, as well as ceramics like zirconia. Extensive investigations have been done in the past decades to understand and model the mechanical behavior of materials during transformation (for example, Sun and Hwang, 1993a, b; Wang, 1997; Sano et al., 1992; Silling, 1992; Stam, 1994; etc.). Here we only mention a few papers, more completed literature may be found in a recent review paper by Ficher et al. (1996). However, most of these investigations are limited to the quasi-static cases.

In fact, a certain incubation time will be necessary for transformation to start, i.e., nucleation, but this is only in the order of hundreds of nanoseconds. Once the transformation has initiated, it will typically proceed throughout the crystal as growth of inclusion with very high speed, even over 1000 m/s. For such a high-speed propagation process of martensite, it is extremely difficult to investigate its nature using an experimental approach. The research work on dynamic analysis of transformation are far few compared with those on quasi-static analysis. One approach to study the propagation behavior of martensitic transformation is to assume that the growth of martensite takes place by the propagation of waves throughout the materials. By assuming the growth velocity according to the velocity of stress waves, one could predict the final morphology of martensitic transformation. For example, Meyers (1980) described the growth of lenticular martensite typically occurring in the Fe-Ni and Fe-C systems in terms of the propagation of transformation waves, and his model could determine the shape of a growing martensite plate. Whereas it is a well-known fact that in reality, only a few martensitic transformations can take place in the order of wave speed. Later Yu and Clapp (1989a) used both the magnetic induction and acoustic emission method to measure the velocity of martensitic growth, and then they (1989b) used molecular dynamics simulations to investigate the growth process. Mikata and Nemat-Nasser (1988) solved the eigenstrain problem wherein the transformation strain is spatially uniform in the inclusion and is time-harmonic, but the boundary of the inclusion is fixed. The key to the growth dynamics of martensitic transformation is the growth velocity of the moving interface and how it is affected by various driving forces and resisting forces in connection with temperature or external load. To the author's knowledge, no widely accepted conclusions exist about it at present stage.

In this paper, the objective is to incorporate the dynamic effect into the propagation process of a transformation inclusion. Based on the propagation equation of a spherical transformation inclusion with pure dilatational eigenstrain, the dynamic elastic fields both inside and outside the inclusion were derived explicitly first. Then following the basic ideas of crack propagation problems in dynamic fracture mechanics (Freund, 1990), the reduction rate of the mechanical part of the free energy accompanying the growth of the transformation inclusion was derived as the driving force for the propagation of the interface. Based on such an energy equation, the equation for determining the propagation speed was established. Although all the explicit expressions obtained are for self-similar growth of a spherical transformation region, some conclusions reveal some general characteristics of the growth stage. And the idea of this paper can also be extended easily to study the growth process of more realistic martensitic morphology.

2 The Stress Fields in an Elastic Solid With a Growing Martensitic Inclusion

2.1 Basic Equations for Dynamic Eigenstrain Problem. Using the convention that repeated suffixes imply summation, the equation of motion can be written in the form as

$$\frac{\partial \sigma_{jk}}{\partial x_k} + f_j = \rho \frac{\partial^2 u_j}{\partial t^2} \tag{1}$$

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where $u_i(\mathbf{x}, t)$ is the total displacement field, ρ is the density of the medium, and f_j is the body-force per unit volume.

Let us consider a body that tends to undergo a nonelastic small deformation $e^0_{jk}(\mathbf{x}, t)$. Suppose that the body occupies a volume D and is bounded by the traction-free surface L, then the stress σ_{jk} is given as

$$\sigma_{jk} = C_{jklm}(e_{lm} - e^0_{lm}) = \tau_{jk} - C_{jklm}e^0_{lm} \tag{2}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the total strain, and $\tau_{jk} = C_{jklm}e_{lm}$ is the stress that would be produced if the displacement $u_i(\mathbf{x}, t)$ were completely elastic. Inserting (2) into the equation of motion with no body forces, we obtain

$$\frac{\partial \tau_{jk}}{\partial x_k} - \rho\frac{\partial^2 u_j}{\partial t^2} = C_{jklm}\frac{\partial e^0_{lm}}{\partial x_k} \tag{3}$$

with the boundary condition that $\tau_{jk}n_k - C_{jklm}n_k e^0_{lm} = 0$ on L. Comparing Eqs. (1) and (3), we see that the nonelastic deformation $e^0_{jk}(\mathbf{x}, t)$ produces the same total displacement as that by a body force $f_j = -C_{jklm}\partial e^0_{lm}/\partial x_k$ throughout D and surface traction $F_j = C_{jklm}n_k e^0_{lm}$ over L. By introducing the dynamic Green's function $G_{ij}(\mathbf{x}' - \mathbf{x};\ t' - t)$, which means the x'_j-component of the displacement produced at $(\mathbf{x}';\ t')$ by a concentrated impulsive unit force in the x_i-direction at $(\mathbf{x}, t)$, one obtains (Willis, 1965; Mura, 1987) for the case without external surface force

$$u_i(\mathbf{x}', t') = \int_{-\infty}^{\infty} dt \int_D C_{jklm}e^0_{lm}(\mathbf{x}, t)\frac{\partial G_{ij}(\mathbf{x}' - \mathbf{x};\ t' - t)}{\partial x_k}dv(\mathbf{x}). \tag{4}$$

If the action of external surface force is considered, Eq. (4) becomes

$$u_i(\mathbf{x}', t') = \int_{-\infty}^{\infty} dt \int_D C_{jklm}e^0_{lm}(\mathbf{x}, t)\frac{\partial G_{ij}(\mathbf{x}' - \mathbf{x};\ t' - t)}{\partial x_k}dv(\mathbf{x}) + \int_{-\infty}^{\infty} dt \int_L F_j(\mathbf{x};\ t)G_{ij}(\mathbf{x}' - \mathbf{x};\ t' - t)ds(\mathbf{x}). \tag{5}$$

2.2 Solution for Spherical Inclusion. Now we consider a transformation inclusion as a region Ω with boundary Γ inside elastic medium D, which undergoes a spontaneous uniform nonelastic deformation e^0_{lm} (Fig. 1).

Considering the growth of the inclusion, the eignstrain is expressed as

$$e^0_{lm} = \epsilon^T_{lm}(t)H(R(t) - |\mathbf{x}|) \tag{6}$$

where $R(t)$ is the radius of the inclusion, and $H(\cdot)$ is the Heaviside's function. For $R(t)$ = constant, it corresponds to Nemat-Nasser's work when ϵ^T_{lm} is time harmonic. Here for simplicity, the current problem involves a time-independent ϵ^T_{lm}, and the self-similar growth of spherical inclusion was considered. If the boundary L is traction free, the total displacement field is derived by inserting Eq. (6) into Eq. (5) and using Gauss's theorem,

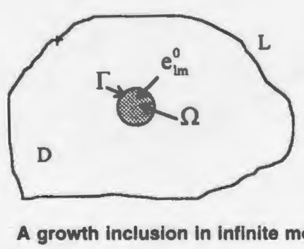

Fig. 1 A growth inclusion in infinite medium

$$u_i(\mathbf{x}', t') = \int_{-\infty}^{\infty} dt \int_{\Omega(R)} C_{jklm}\epsilon^T_{lm}H(R(t) - |\mathbf{x}|) \times \frac{\partial G_{ij}(\mathbf{x}' - \mathbf{x};\ t' - t)}{\partial x_k}dv(\mathbf{x})$$
$$= \int_{-\infty}^{\infty} dt \int_{\Gamma} C_{jklm}\epsilon^T_{lm}n_k G_{ij}(\mathbf{x}' - \mathbf{x};\ t' - t)ds(\mathbf{x}), \tag{7}$$

where Γ is the surface of the spherical inclusion and n_k is the outward normal to the surface.

In an isotropic medium, the appropriate expression for dynamic Green's function $G_{ij}(\mathbf{x};\ t)$ has been given by Love (1944) as follows:

$$G_{ij}(\mathbf{X};\ t) = \frac{1}{4\pi\rho}\left\{\frac{\delta(x - bt)}{bx}\delta_{ij} + \frac{\partial^2 x^{-1}}{\partial x_i \partial x_j}t + \frac{x_i x_j}{x^3}\left[\frac{\delta(x - at)}{a} - \frac{\delta(x - bt)}{b}\right]\right\} \quad 0 \le bt \le x \le at$$
$$= 0 \quad \text{otherwise} \tag{8}$$

where $\delta(\cdot)$ is Dirac's delta function, $a^2 = (\lambda + 2\mu)/\rho$, $b^2 = \mu/\rho$ and $x^2 = x_j x_j$, and λ, μ are Lame's constants.

We consider the spherical transformation inclusion undergoing a uniform dilatation ϵ^T throughout its volume, and having the same elastic moduli as the matrix. Hence

$$\epsilon^T_{lm} = \epsilon^T\delta_{lm}. \tag{9}$$

Substitution of Eq. (9) into Eq. (7) yields

$$u_i(\mathbf{x}', t') = (3\lambda + 2\mu)\int_{-\infty}^{\infty} dt \int_{S(R)} \epsilon^T\frac{x_j}{R(t)}G_{ij}(\mathbf{x}' - \mathbf{x};\ t' - t)ds(\mathbf{x}). \tag{10}$$

By symmetry, in polar coordinates (r, θ, ϕ), the only nonzero displacement is $u_r(r, t')$, which coincides with $u_3(\mathbf{x}';\ t')$ at any point $\mathbf{x}' = (0, 0, r)$ of the x'_3-axis. Therefore,

$$u_r(r, t') = u_3(0, 0, r;\ t') = (3\lambda + 2\mu)\int_{-\infty}^{\infty} dt \int_{S(R)} \frac{x_j}{R(t)}G_{3j}(\mathbf{x}' - \mathbf{x};\ t' - t)ds(\mathbf{x})$$
$$= \frac{3\lambda + 2\mu}{2\rho}\int_{-\infty}^{\infty} dt \int_0^{\pi}\left\{\frac{\cos\theta\,\delta(x - bt)}{bx} + \frac{1}{x^3}[3(r\cos\theta - R)(r - R\cos\theta) - x^2\cos\theta] + \frac{(r\cos\theta - R)(r - R\cos\theta)}{x^3}\left[\frac{\delta(x - at)}{a} - \frac{\delta(x - bt)}{b}\right]\right\}R^2(t)\sin\theta d\theta \tag{11}$$

where we have substituted for G_{3j} from Eq. (8) with $\mathbf{x}' = (0, 0, r)$ and $\mathbf{x} = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$, and performed the trivial integration with respect to ϕ. The θ-integration can be carried out by transforming the independent variable to x as follows:

$$\cos\theta = \frac{R^2(t) + r^2 - x^2}{2R(t)r} \tag{12}$$

and the integration range for θ is $(0, \pi)$, which should change to $(|r - R|, r + R)$ for x. By considering the nonzero range of G_{3j} given by Eq. (8), the integration range should be (bt, at) for x. Through the integration, one can find that the components with speed b canceled each other. One obtains

$$u_r(r, t') = u_3(0, 0, r; t') = \frac{3\lambda + 2\mu}{2\rho} \cdot \frac{\epsilon^T}{a} \int_{t_L}^{t_U} \psi(t'; t)dt \tag{13}$$

where (t_L, t_U) are the lower and upper limits of the integration and are determined by $|r - R(t)| \le a(t' - t) \le r + R(t)$, and

$$\psi(t'; t) = \begin{cases} \dfrac{r^2 + R^2(t) - a^2(t' - t)^2}{2r^2}; \\ \qquad |r - R(t)| \le a(t' - t) \le r + R(t) \\ 0; \quad \text{otherwise.} \end{cases} \tag{14}$$

If we know the propagation equation $R(t)$ of the transformation inclusion, we can derive the dynamical displacement field both inside and outside the inclusion through Eq. (13).

2.3 Spherical Inclusion With Constant Propagation Velocity

(1) Interior Points. If the point $\mathbf{x}$ is inside the inclusion ($r - R(t) < 0$ or $r < R(t)$), equations for determining the upper and lower limits of the integration variable t are as follows:

$$a(t' - t_U^I) - R(t_U^I) + r \ge 0 \tag{15}$$

$$a(t' - t_L) - R(t_L) - r \le 0. \tag{16}$$

If we further assume that the inclusion expands at constant speed v, $R(t) = vt$. Equations (15) and (16) give the integration range

$$\frac{at' - r}{a + v} \le t \le \frac{at' + r}{a + v}. \tag{17}$$

Carrying out the integration (13) on the range (17) gives radial displacement inside the inclusion as follows:

$$u_r^I(t) = \frac{3\lambda + 2\mu}{3\rho a(a + v)^2}(a + 2v)r\epsilon^T \tag{18}$$

The total strain inside the inclusion is given by

$$\epsilon_r = \frac{\partial u_r}{\partial r} = \epsilon_t = \frac{u_r}{r} = \frac{3\lambda + 2\mu}{3\rho a(a + v)^2}(a + 2v)\epsilon^T. \tag{19}$$

For isotropic materials, the stress inside the inclusion can be obtained as follows:

$$\begin{aligned} \sigma_{ij}^I &= C_{ijkl}(\epsilon_{kl}^I - \epsilon_{kl}^T) \\ &= [\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk}](\epsilon_r - \epsilon^T)\delta_{kl} \\ &= \left[\frac{1}{\left(1 + \dfrac{4\mu}{3\lambda + 2\mu}\right)\left(1 + \dfrac{v^2}{a(a + v)}\right)} - 1\right] \\ &\qquad \times (3\lambda + 2\mu)\epsilon^T\delta_{ij}. \end{aligned} \tag{20}$$

It is interesting to note that inside an expanding inclusion at a constant velocity, the elastic field is time-independent and the strain and stress are constant for uniform eignstrain as the case of a static inclusion problem (Mura, 1987). If the growth speed v approaches zero, Eqs. (18) and (19) approaches the result of the static inclusion problem by substituting $a^2 = (\lambda + 2\mu)/\rho$ into them. It can be found through Eq. (20) that if the eigenstrain of the inclusion is extension, the stresses inside the spherical inclusion are always hydrostatic compressive as expected. Further, the compressive stress will increase with the increasing speed of inclusion expansion. If the growth speed of the inclusion reaches the speed a of irrotational waves, the total extension strain will be reduced 25 percent as compared with the total strain of the static case (see Eq. (19)), and the hydrostatic compressive stress inside inclusion increases by 25 percent as given by Eq. (20).

(2) Exterior Points. In such case, the equation for determining the upper and lower limits of the integration is given by

$$a(t' - t_U^O) + R(t_U^O) - r \ge 0 \tag{21}$$

$$a(t' - t_L) - R(t_L) - r \le 0. \tag{22}$$

For uniform expansion inclusion, the integration range is given by

$$\frac{at' - r}{a + v} \le t \le \frac{at' - r}{a - v}. \tag{23}$$

Substitution of Eq. (23) into Eq. (13) gives

$$\begin{aligned} u_r^O(t) = \frac{(3\lambda + 2\mu)\epsilon^T}{4\rho a r^2(a^2 - v^2)^3} \Big\{ & 2vr^2(at - r)(a^2 - v^2)^2 \\ & + \frac{2}{3}v^3(at - r)^3(3a^2 + v^2) + \frac{1}{3}a^2[(r - vt)^3(a + v)^3 \\ & - (r + vt)^3(a - v)^3] \Big\} \end{aligned} \tag{24}$$

$$\sigma_{rr}^O = (\lambda + 2\mu)\frac{\partial u_r^O}{\partial r} + 2\lambda\frac{u_r^O}{r}. \tag{25}$$

One can check easily that on the surface of the inclusion (i.e., $r = vt$)

$$u_r^I(t) = u_r^O(t) \tag{26}$$

which satisfies the continuity condition on the surface.

3 Energy Concepts and the Equation of Martensitic Inclusion Growth

Accompanying the propagation of the transformation inclusion, the energy will change. Similar to the dynamic fracture process (Freund, 1990), the released energy serves as a generalized driving force for the growth of the inclusion. In this section, we will derive the expression of the energy release rate, and establish the equation to determine the growth rate of the inclusion.

3.1 The Rate of Mechanical Energy. Consider a finite body D containing a transformation inclusion Ω. Along the surface S of D, the external force is $T_i = \sigma_{ij}n_j$, and the inner and outer surfaces of Ω are denoted as Γ^-, Γ^+ (Fig. 2).

The mechanical part of the internal energy involves the elastic energy plus the kinetic energy. Therefore, the rate of elastic strain energy inside D is given by

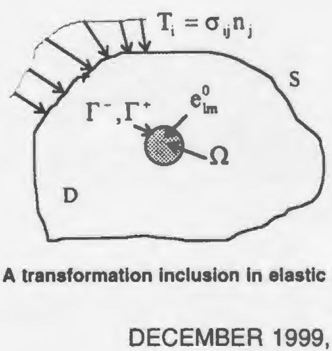

Fig. 2 A transformation inclusion in elastic medium

$$\dot{W} = \frac{d}{dt}\left[\frac{1}{2}\int_D \sigma_{ij}e_{ij}dv\right] \tag{27}$$

where

$$\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \epsilon^T_{kl}) \tag{28}$$

$$e_{ij} = \epsilon_{ij} - \epsilon^T_{ij}. \tag{29}$$

Substitution of Eqs. (28), (29) into Eq. (27) yields

$$\begin{aligned}\dot{W} &= \frac{1}{2}\int_D \frac{d}{dt}\{\sigma_{ij}(\epsilon_{ij} - \epsilon^T_{ij})\}dv\\ &= \frac{1}{2}\int_D \frac{d}{dt}\left\{C_{ijkl}\left(\frac{\partial u_i}{\partial x_j} - \epsilon^T_{ij}\right)\left(\frac{\partial u_k}{\partial x_l} - \epsilon^T_{kl}\right)\right\}dv\\ &= \int_D C_{ijkl}\left(\frac{\partial \dot{u}_i}{\partial x_j} - \dot{\epsilon}^T_{ij}\right)\left(\frac{\partial u_k}{\partial x_l} - \epsilon^T_{kl}\right)\Bigg\}dv\\ &= \int_D \sigma_{ij}\left(\frac{\partial \dot{u}_i}{\partial x_j} - \dot{\epsilon}^T_{ij}\right)dv.\end{aligned} \tag{30}$$

By substituting Eq. (1) into Eq. (30), the first term of Eq. (30) can be written as

$$\begin{aligned}\int_D \sigma_{ij}\frac{\partial \dot{u}_i}{\partial x_j}dv &= \int_D \frac{\partial}{\partial x_j}(\sigma_{ij}\dot{u}_i)dv - \int_D \sigma_{ij,j}\dot{u}_i dv\\ &= \int_D \frac{\partial}{\partial x_j}(\sigma_{ij}\dot{u}_i)dv - \int_D \rho\ddot{u}_i\dot{u}_i dv.\end{aligned} \tag{31}$$

In an elastic medium with a growing inclusion, we cannot use Gauss's theorem directly to the first term in Eq. (31), since across the boundary of the inclusion, the integrand is not continuous. So we divide the integral into two parts as follows:

$$\begin{aligned}\int_D \frac{\partial}{\partial x_j}(\sigma_{ij}\dot{u}_i)dv &= \int_\Omega \frac{\partial}{\partial x_j}(\sigma_{ij}\dot{u}_i)dv + \int_{D-\Omega}\frac{\partial}{\partial x_j}(\sigma_{ij}\dot{u}_i)dv\\ &= \int_{\Gamma^-}\sigma_{ij}\dot{u}_i n_j ds - \int_{\Gamma^+}\sigma_{ij}\dot{u}_i n_j ds\\ &\quad + \int_S \sigma_{ij}\dot{u}_i n_j ds\\ &= -\int_\Gamma [\sigma_{ij}\dot{u}_i]n_j ds + \int_S \sigma_{ij}\dot{u}_i n_j ds\end{aligned} \tag{32}$$

where Γ is the surface of the inclusion, n_j is the exterior unit vector normal to Γ, and

$$[\sigma_{ij}\dot{u}_i] = (\sigma_{ij}\dot{u}_i)_{\text{out}} - (\sigma_{ij}\dot{u}_i)_{\text{in}}. \tag{33}$$

The change rate of the kinetic energy is given by

$$\dot{K} = \int_D \rho\dot{u}_i\ddot{u}_i dv. \tag{34}$$

Therefore the change rate of the internal energy is given by

$$\begin{aligned}\dot{E} &= \dot{W} + \dot{K}\\ &= -\int_\Gamma [\sigma_{ij}\dot{u}_i]n_j ds - \int_D \sigma_{ij}\dot{\epsilon}^T_{ij}dv + \int_S \sigma_{ij}\dot{u}_i n_j ds.\end{aligned} \tag{35}$$

The rate of work done by the external force T_i is given by

$$U = \int_S T_i\dot{u}_i ds = \int_S \sigma_{ij}n_j\dot{u}_i ds. \tag{36}$$

Therefore the energy release rate accompanying the growth of the transformation inclusion can be obtained as follows:

$$\dot{\Pi} = U - \dot{E} = \int_\Gamma [\sigma_{ij}\dot{u}_i]n_j ds + \int_\Omega \sigma_{ij}\dot{\epsilon}^T_{ij}dv. \tag{37}$$

According to Eqs. (6) and (9), one can obtain

$$\dot{\epsilon}^0_{ij} = \epsilon^T\delta_{ij}\delta(R(t) - |x|)\dot{R}(t). \tag{38}$$

The volume integral in Eq. (37) can be carried out first on $|x|$, and one obtains

$$\dot{\Pi} = \int_\Gamma \{[\sigma_{ij}\dot{u}_i]n_j + \sigma_{ii}\epsilon^T\dot{R}(t)\}ds. \tag{39}$$

Eshelby proposed the famous energy-momentum tensor as the driving force for a defect to propagate in materials (Eshelby, 1970). In fact, Eq. (39) gives the rate of the driving force for a transformation region to expand in the same sense. $\dot{\Pi}$ also means the reduction rate of the mechanical part of the free energy accompanying the growth of the transformation inclusion. For spherical inclusion with pure dilatational eigenstrain as discussed in this paper, Eq. (39) becomes

$$\begin{aligned}\dot{\Pi} &= \int_\Gamma \{[\sigma_{ij}\dot{u}_i]n_j + \sigma_{ii}\epsilon^T\dot{R}(t)\}ds\\ &= S(R)\{[\sigma_{rr}\dot{u}_r] + \sigma_{ii}\epsilon^T\dot{R}(t)\}\\ &= S(R)\Bigg\{(\lambda + 2\mu)\left[\frac{\partial u^O_r}{\partial r}\dot{u}^O_r - \left(\frac{\partial u^I_r}{\partial r} - \epsilon^T\right)\dot{u}^I_r\right]\\ &\quad + 2\lambda\left[\frac{u^O_r}{r}\dot{u}^O_r - \left(\frac{u^I_r}{r} - \epsilon^T\right)\dot{u}^I_r\right]\\ &\quad + \epsilon^T\dot{R}(t')\Bigg[\sigma^0_{ii} + (3\lambda + 2\mu)\\ &\qquad \times\left(\frac{\partial u^I_r}{\partial r} + 2\frac{u^I_r}{r} - 3\epsilon^T\right)\Bigg]\Bigg\}\Bigg|\, r = R(t')\end{aligned} \tag{40}$$

where $S(R)$ is the surface area of the inclusion. The radial displacements u^I_r, u^O_r inside and outside the inclusion are given by Eq. (13) as follows:

$$u^I_r = C\int_{t_L}^{t^I_u} \psi(t'; t)dt \tag{41}$$

$$u^O_r = C\int_{t_L}^{t^O_u} \psi(t'; t)dt \tag{42}$$

where $C = (3\lambda + 2\mu)\epsilon^T/2\rho a$, $\psi(t'; t)$ is given by Eq. (14), and t_L, t^I_U, t^O_U can be determined by Eqs. (15), (16), and (21), respectively. It can be found easily that

$$\left\{ t_U^I = t_U^O = t';\ \frac{\partial t_U^I}{\partial t'} = \frac{\partial t_U^O}{\partial t'} = 1 \right\} \Bigg| r = R(t'). \quad (43)$$

Therefore

$$\dot{u}_r^O = \dot{u}_r^I = C\left[\int_{t_L}^{t'} \frac{\partial \psi(t';t)}{\partial t'} dt + 1 - \psi(t';t_L)\frac{\partial t_L}{\partial t'}\right] \quad (44)$$

$$\frac{\partial u_r^O}{\partial r} = C\left[\int_{t_L}^{t'} \frac{\partial \psi(t';t)}{\partial r} dt + \frac{\partial t_U^O}{\partial r} - \psi(t';t_L)\frac{\partial t_L}{\partial r}\right] \quad (45)$$

$$\frac{\partial u_r^I}{\partial r} = C\left[\int_{t_L}^{t'} \frac{\partial \psi(t';t)}{\partial r} dt + \frac{\partial t_U^I}{\partial r} - \psi(t';t_L)\frac{\partial t_L}{\partial r}\right]. \quad (46)$$

Substitution of Eqs. (44), (45), and (46) into Eq. (40) yields

$$\begin{aligned}\dot{\Pi} = S(R)\Bigg\{ & C\left[(3\lambda + 2\mu)\epsilon^T - \frac{2aC}{a^2 - \dot{R}^2(t')}(\lambda + 2\mu)\right] \\ & \times \left[\int_{t_L}^{t'} \frac{\partial \psi(t';t)}{\partial t'} dt + 1 - \psi(t';t_L)\frac{\partial t_L}{\partial t'}\right] + \epsilon^T \dot{R}(t')\Bigg\{\sigma_{ii}^0 \\ & - 3(3\lambda + 2\mu)\epsilon^T + (3\lambda + 2\mu)C\Bigg[\int_{t_L}^{t'} \frac{\partial \psi(t';t)}{\partial r} dt + \frac{\partial t_U^I}{\partial r} \\ & - \psi(t';t_L)\frac{\partial t_L}{\partial r} + 2\int_{t_L}^{t'} \psi(t';t)dt/R(t')\Bigg]\Bigg\}\Bigg\}. \end{aligned} \quad (47)$$

$\dot{\Pi}$ can also be expressed in the form as

$$\dot{\Pi} = g\dot{R}(t) \quad (48)$$

where $g = \partial \Pi / \partial R$, is the rate of mechanical energy reduction per unit increase of the inclusion radius.

3.2 The Rate of Chemical and Surface Free Energy. If the reduction of surface energy per unit area and the reduction of the chemical free energy per unit volume are denoted as ΔU_s, $\Delta U_{ch} = -\Delta S(T_0 - T)$, where ΔS is the transformational entropy change, T is the temperature (Wang, 1997), and the total free energy release rate per unit increase of R is given by

$$\begin{aligned} G &= g + 4\pi R^2 \Delta U_{ch} + 8\pi R \Delta U_s \\ &= \frac{\dot{\Pi}}{\dot{R}} - 4\pi R^2 \Delta S(T_0 - T) + 8\pi R \Delta U_s. \end{aligned} \quad (49)$$

Considering that the energy will be dissipated in the process of the transformation, one can establish the growth criterion of the transformation region as follows:

$$G = G_c \quad (50)$$

where G_c is the material constant, which represents the dissipated part of the energy. Through Eq. (50), one can obtain the radius of the transformation inclusion as a function of time under different external conditions.

4 Discussion and Some Concluding Remarks

If we take the assumption that after time t_0, the inclusion will grow in constant speed V, therefore, from t_L to t', it expands also in the constant speed, and one obtains

$$t_L = \frac{a - V}{a + V} t' - \frac{2[R(t_0) - Vt_0]}{a + V} \quad (51)$$

$$R(t) = R(t_0) + V(t - t_0). \quad (52)$$

Substitution of Eqs. (51), (52) into Eq. (13), then into Eq. (47) yields

$$\begin{aligned}\dot{\Pi} = S(R)\Bigg\{ & C\left[(3\lambda + 2\mu)\epsilon^T - \frac{2aC}{a^2 - V^2}(\lambda + 2\mu)\right]\Bigg[\frac{R^3(t_L)}{3R^3(t')} \\ & + \frac{8a^2V}{3(a+V)^3} - \frac{2a^2}{(a+V)^2} + \frac{2}{3} - \left(\frac{R^2(t_L)}{2R^2(t')} \right. \\ & \left. - \frac{2a^2}{(a+V)^2} + \frac{1}{2}\right)\frac{a - V}{a + V}\Bigg] + \epsilon^T V\Bigg\{\sigma_{ii}^0 - 3(3\lambda + 2\mu)\epsilon^T \\ & + \frac{(3\lambda + 2\mu)C}{a + V}\left[\frac{R^2(t_L)}{2R^2(t')} - \frac{2a^2}{(a+V)^2} + \frac{7}{2}\right]\Bigg\}\Bigg\}. \end{aligned} \quad (53)$$

Substitution of Eq. (53) into Eq. (49), then into Eq. (50) yields an equation for determining the speed V of the inclusion propagation. It can be found that the speed depends on the time t' through the terms involving $R(t_L)/R(t')$. When the time $t' \to \infty$, one finds

$$\lim_{t' \to \infty} \frac{R(t_L)}{R(t')} = \frac{a - V}{a + V}. \quad (54)$$

Substitution of Eq. (54) into Eq. (53) gives

$$\begin{aligned}\dot{\Pi} = S(R)\Bigg\{ & C\left[(3\lambda + 2\mu)\epsilon^T - \frac{2aC}{a^2 - V^2}(\lambda + 2\mu)\right] \\ & \times \Bigg[\frac{(a-V)^3}{3(a+V)^3} + \frac{8a^2V}{3(a+V)^3} - \frac{2a^2}{(a+V)^2} + \frac{2}{3} \\ & - \left(\frac{(a-V)^2}{2(a+V)^2} - \frac{2a^2}{(a+V)^2} + \frac{1}{2}\right)\frac{a - V}{a + V}\Bigg] \\ & + \epsilon^T V\Bigg\{\sigma_{ii}^0 - 3(3\lambda + 2\mu)\epsilon^T + \frac{(3\lambda + 2\mu)C}{a + V} \\ & \left[\frac{(a-V)^2}{2(a+V)^2} - \frac{2a^2}{(a+V)^2} + \frac{7}{2}\right]\Bigg\}\Bigg\}. \end{aligned} \quad (55)$$

Substitution of Eq. (55) into Eq. (49), then into Eq. (50) yields an equation for determining the steady speed of the inclusion expansion when $t' \to \infty$.

If it is assumed that the boundary of the transformation inclusion grows in constant speed V, one can write

$$t_L = \frac{a - V}{a + V} t',\ R(t) = Vt. \quad (56)$$

Therefore, from Eq. (47), one can find that reduction rate of the mechanical part of the free energy $\dot{\Pi}$ is also given by Eq. (55). Substitution of Eq. (55) into Eq. (49), then into Eq. (50) yields

$$\begin{aligned} \frac{4\pi R^2}{V}\Bigg\{ & C\left[(3\lambda + 2\mu)\epsilon^T - \frac{2aC}{a^2 - V^2}(\lambda + 2\mu)\right]\Bigg[\frac{(a-V)^3}{3(a+V)^3} \\ & + \frac{8a^2V}{3(a+V)^3} - \frac{2a^2}{(a+V)^2} + \frac{2}{3} - \left(\frac{(a-V)^2}{2(a+V)^2} \right. \\ & \left. - \frac{2a^2}{(a+V)^2} + \frac{1}{2}\right)\frac{a - V}{a + V}\Bigg] + \epsilon^T V\Bigg\{\sigma_{ii}^0 - 3(3\lambda + 2\mu)\epsilon^T \\ & + \frac{(3\lambda + 2\mu)C}{a + V}\left[\frac{(a-V)^2}{2(a+V)^2} - \frac{2a^2}{(a+V)^2} + \frac{7}{2}\right]\Bigg\}\Bigg\} \\ & - 4\pi R^2 \Delta S(T_0 - T) + 8\pi R \Delta U_s = G_c. \end{aligned} \quad (57)$$

Using Eq. (57), one can determine the constant growth speed of the transformation inclusion according to the applied environment temperature and hydrostatic load for different materials. The dissipated part of the energy can be assumed to be proportional to the volume of the inclusion, therefore G_c can be expressed in the form as

$$G_c = 4\pi R^2 \xi_c \tag{58}$$

where ξ_c is the dissipated energy for creating unit volume of martensite. Substitution of Eq. (58) into Eq. (57) yields

$$\frac{1}{V}\left\{C\left[(3\lambda+2\mu)\epsilon^T - \frac{2aC}{a^2-V^2}(\lambda+2\mu)\right]\left[\frac{(a-V)^3}{3(a+V)^3} + \frac{8a^2V}{3(a+V)^3} - \frac{2a^2}{(a+V)^2} + \frac{2}{3} - \left(\frac{(a-V)^2}{2(a+V)^2} - \frac{2a^2}{(a+V)^2} + \frac{1}{2}\right)\frac{a-V}{a+V}\right] + \epsilon^T V\left\{\sigma_{ii}^0 - 3(3\lambda+2\mu)\epsilon^T + \frac{(3\lambda+2\mu)C}{a+V}\left[\frac{(a-V)^2}{2(a+V)^2} - \frac{2a^2}{(a+V)^2} + \frac{7}{2}\right]\right\}\right\} - \Delta S(T_0 - T) + \frac{2}{R}\Delta U_s = \xi_c. \tag{59}$$

From Eq. (59), we know, if the surface energy ΔU_s can be neglected, under the action of constant hydrostatic tension, or constant temperature, the interface of martensitic inclusion can grow in constant speed, whereas if the surface energy cannot be neglected, to obtain the constant speed for martensite growth, the applied external condition must change in proportion to $\propto 1/R$.

If we further assume that the reduction of the chemical free energy compensates for the dissipation energy in the transformation process, i.e., $\Delta U_{ch} = -\Delta S(T_0 - T) = \xi_c$, one obtains

$$\frac{1}{V}\left\{C\left[(3\lambda+2\mu)\epsilon^T - \frac{2aC}{a^2-V^2}(\lambda+2\mu)\right]\left[\frac{(a-V)^3}{3(a+V)^3} + \frac{8a^2V}{3(a+V)^3} - \frac{2a^2}{(a+V)^2} + \frac{2}{3} - \left(\frac{(a-V)^2}{2(a+V)^2} - \frac{2a^2}{(a+V)^2} + \frac{1}{2}\right)\frac{a-V}{a+V}\right] + \epsilon^T V\left\{\sigma_{ii}^0 - 3(3\lambda+2\mu)\epsilon^T + \frac{(3\lambda+2\mu)C}{a+V}\left[\frac{(a-V)^2}{2(a+V)^2} - \frac{2a^2}{(a+V)^2} + \frac{7}{2}\right]\right\}\right\} + \frac{2}{R}\Delta U_s = 0. \tag{60}$$

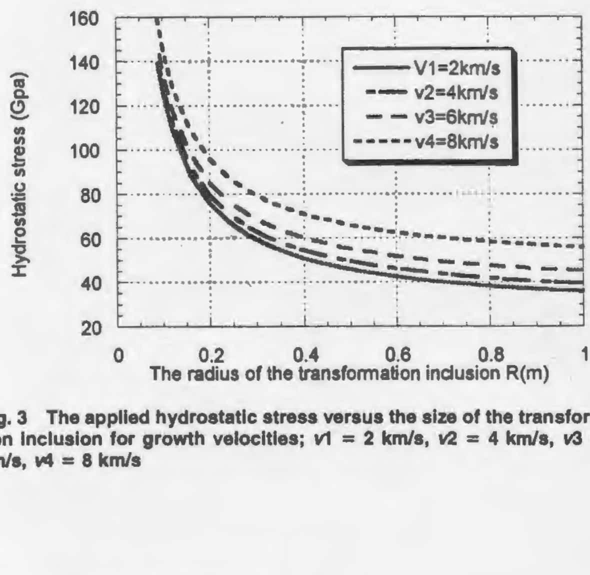

Fig. 3 The applied hydrostatic stress versus the size of the transformation inclusion for growth velocities; $v1$ = 2 km/s, $v2$ = 4 km/s, $v3$ = 6 km/s, $v4$ = 8 km/s

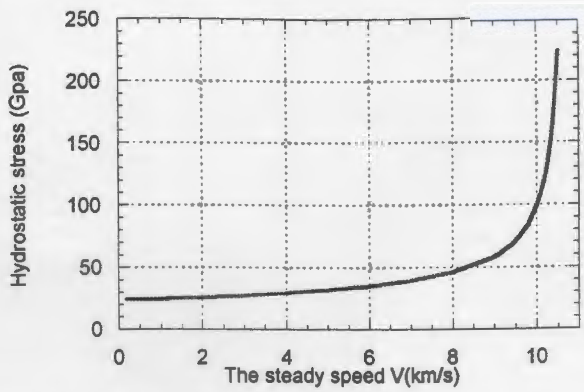

Fig. 4 The applied hydrostatic stress versus steady speed of the inclusion expansion

Using Eq. (60), one can derive σ_{ii}^0 required to keep the interface propagating in constant speed as a function of the inclusion size. If we take the ceramic material as an example, whose constants are $\lambda = 130.77$ Gpa, $\mu = 80.15$, $\rho = 2500$ kg/m^3, $\epsilon^T = 0.04$, $\Delta U_s = -200$ Mpa, the applied hydrostatic stress was shown in Fig. 3. The results shown here do not correspond to any practical case.

The applied hydrostatic stress versus the steady speed was shown in Fig. 4. It is obvious that the growth speed is always lower than the speed $a = 10790$ m/s of irrotational waves.

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