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Investigation of the scattering of harmonic elastic waves by a finite crack using the non-local theory

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1. Introduction

In several previous papers [1,2,3,4], Eringen had discussed the state of stress near the tip of a sharp line crack in an elastic plate subject to uniform tension, shear and anti-plane shear. The field equations employed in the solution of these problems are those of the theory of non-local elasticity. The solutions obtained did not contain any stress singularity, thus resolving a fundamental problem that persisted over many years. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way, and also the non-local elasticity has a big potential to understand the behavior of composite materials. It is of interest to note that applications of the non-local theory to concrete problems often lead to impressive agreements with the data of experiments and observations (e.g. [5,6]). And in papers [7,8], they discussed the propagation of

Love wave, the wave propagation in elastic plate by use of non-local theory. However, analytical treatment on the dynamic crack problem by use the non-local theory has not been attempted.

The present paper deals with the dynamic problem of a line crack in an elastic plate where the crack surface is subjected to the harmonic elastic wave. The field equations of non-local elasticity theory were employed to formulate and solve this problem. In solving the equations, the crack surface displacement is expanded in a series using Jacobi's polynomials and Schmidt's method is used. The solution, as expected, does not contain the stress singularity near the crack tips.

2. Basic Equations of Non-local Elasticity

Basic equations of linear, homogeneous, isotropic, and elastic solids, for a non-local theory of elasticity are given by

$$\tau_{kl,k} = \rho \ddot{u}_l \tag{1}$$

$$\tau_{kl} = \int_V [\lambda'(|X'-X|)e_{rr}(X',t)\delta_{kl} + 2\mu'(|X'-X|)e_{kl}(X',t)]dV' \tag{2}$$

$$e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{3}$$

where the only difference from classical elasticity is in the stress constitutive equation (2) in which the stress $\tau_{kl}(X)$ at a point X depends on the strains $e_{kl}(X')$, at all points of the body. For homogeneous and isotropic solids there exist only two material constants, $\lambda'(|X'-X|)$ and $\mu'(|X'-X|)$ which are functions of the distance $|X'-X|$. The integral in equation (2) is over the volume V of the body enclosed within a surface ∂V. λ' and μ' can be written as follows [3,4,9]:

$$(\lambda', \mu') = (\lambda, \mu)\alpha(|X'-X|) \tag{4}$$

$\alpha(|X'-X|)$ is known as influence function, and is the function of the distance $|X'-X|$. λ and μ are the Lame constants of classical elasticity. ρ is the mass density of the material.

Substitution of equation (4) into equation (2) yields

$$\tau_{kl}(X,t) = \int_V \alpha(|X'-X|)\sigma_{kl}(X',t)dV(X') \tag{5}$$

where $\sigma_{ij}(X',t) = \lambda e_{rr}(X',t)\delta_{ij} + 2\mu e_{ij}(X',t) = \lambda u_{r,r}(X',t)\delta_{ij} + \mu[u_{i,j}(X',t) + u_{j,i}(X',t)]$ (6)

The expression of equation (6) is the classical Hook's law. Substituting equation (6) into equation (1) and using Green-Gauss theorem, it can be shown

$$\int_V \alpha(|X'-X|)[(\lambda+\mu)u'_{k,kl}(X',t)+\mu u'_{l,kk}(X',t)]dV(X')$$
$$-\int_{\partial V}\alpha(|X'-X|)\sigma_{kl}(X',t)da_k(X') = \rho\ddot{u}_l \qquad (7)$$

Here the surface integral may be dropped if the only surface of the body is at infinity.

3. The Crack Model

It is assumed that there is a line crack in an elastic plate as shown in Fig.1. Let ω be the circular frequency of the incident wave. In what follows, the time dependence of all field quantities assumed to be of the form $e^{-i\omega t}$ will be suppressed but understood. It was further supposed that the two faces of the crack do not come in contact during vibrations. When the crack is subjected to the harmonic elastic waves, as discussed in [10] the boundary conditions on the crack faces at $y=0$ are

$$\tau_{yx}(x,0,t)=0 \quad , \quad v(x,0,t)=0, \qquad |x|\rangle l \qquad (8)$$

$$\tau_{yx}(x,0,t)=0 \quad , \quad \tau_{yy}(x,0,t)=-\tau_0 \, , \qquad |x|\le l \qquad (9)$$

$$u(x,y,t)=v(x,y,t)=0 \qquad , \quad (x^2+y^2)^{1/2}\to\infty \qquad (10)$$

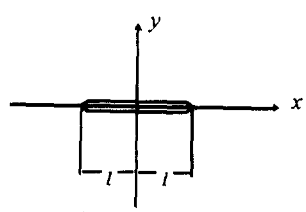

Fig.1. Incidence of a time harmonic wave on the crack of the length $2l$

In this paper, the wave is vertically incident and we only consider that τ_0 is positive.

4. The Dual Integral Equations

According to the boundary conditions, the equation (7) can be written as follow:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(|x'-x|,|y'-y|)[(\lambda+\mu)u'_{k,kj}(x',y',t)+\mu u'_{j,kk}(x',y',t)]dx'dy'$$

$$-2\mu\int_{-l}^{l}\alpha(|x'-x|,|y|)\,[\![e_{2j}(x',0,t)]\!]\,dx' = -\rho\omega^2 u_j \tag{11}$$

where $[\![e_{2j}(x',0,t)]\!] = e_{2j}(x',0^+,t)-e_{2j}(x',0^-,t)$ is a jump across the crack.

$$e_{kj}(x,y,t)=\frac{1}{2}[u_{k,j}(x,y,t)+u_{j,k}(x,y,t)]$$

From the reference [2], it was showen

$$[\![e_{2j}(x,0,t)]\!] = 0 \qquad \text{for all } x \tag{12}$$

Define the Fourier transform by the equations

$$\bar{f}(s)=\int_{-\infty}^{\infty}f(x)e^{-isx}dx \tag{13}$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(s)e^{isx}ds \tag{14}$$

For solving the problem, the Fourier transform of equation (11) with respect x can be given as follows:

$$\int_{-\infty}^{\infty}\bar{\alpha}(|s|,|y'-y|)[\mu\bar{u}'_{,yy}-(\lambda+2\mu)s^2\bar{u}'-is(\lambda+\mu)\bar{v}'_{,y}]dy'=-\rho\omega^2\bar{u} \tag{15}$$

$$\int_{-\infty}^{\infty}\bar{\alpha}(|s|,|y'-y|)[-is(\lambda+\mu)\bar{u}'_{,y}+(\lambda+2\mu)\bar{v}'_{,yy}-s^2\mu\bar{v}']dy'=-\rho\omega^2\bar{v} \tag{16}$$

For the influence function α, it seems obvious that one has to resort to an approximate procedure. In the given problem, the appropriate numerical procedure seems to spring quite naturally from the hypothesis of the attenuating neighborhood underlying the theory of non-local continua. According to this hypothesis, the influence of the particle of the body, on the thermoelectric state at the particle under observation, subsides rather rapidly with an increasing distance from particle. In the classical theory, the function that characterizes the particle interactions is the Dirac delta function since in this theory the actions are assumed to has a zero range. In non-local theories the intermolecular forces may be represented by a variety of functions as long as their values decrease rapidly with the distance. In the present study, as adequate functions we decide to select the terms, $\delta_n(y'-y)$, n=1,2,..., of the so call δ-sequences. A δ-sequence, as generally known, is (in the present case a one-dimensional) Dirac delta function, $\delta(y'-y)$. With respect to the terms of the adopted delta sequence we accept the following simplifying assumptions: (see the paper [8])

(a) : For a sufficiently large b (as compared with the sphere of interactions of the particles), it is permissible to make the replacement

$$\int_{-b}^{b} f(y')\delta_n(y'-y)dy' \approx \int_{-\infty}^{\infty} f(y')\delta(y'-y)dy' \tag{17}$$

(b): As a consequence of the foregoing, the terms $\delta_n(y'-y), n\rangle\rangle 1$,acquire the shifting properly characteristic of the Dirac function,

$$\int_{-b}^{b} f(y)\delta_n(y-y)dy \approx f(y) \tag{18}$$

We now set

$$\overline{\alpha}(|s|,|y'-y|) = \overline{\alpha}_0(s)\delta_n(y'-y) \tag{19}$$

From the equations (15) and (16), it can be obtained

$$\overline{\alpha}_0(s)[\mu\overline{u},_{yy}-(\lambda+2\mu)s^2\overline{u}-is(\lambda+\mu)\overline{v},_y] = -\rho\omega^2\overline{u} \tag{20}$$

$$\overline{\alpha}_0(s)[-is(\lambda+\mu)\overline{u},_y+(\lambda+2\mu)\overline{v},_{yy}-s^2\mu\overline{v}] = -\rho\omega^2\overline{v} \tag{21}$$

whose solutions do not present difficulties, we have($y \geq 0$)

$$u(x,y,t) = -\frac{2}{\pi}\int_0^\infty sA_1(s)\sin(sx)\exp(-\gamma_1 y)ds - \frac{2}{\pi}\int_0^\infty \gamma_2 A_2(s)\sin(sx)\exp(-\gamma_2 y)ds \tag{22}$$

$$v(x,y,t) = -\frac{2}{\pi}\int_0^\infty \gamma_1 A_1(s)\cos(sx)\exp(-\gamma_1 y)ds - \frac{2}{\pi}\int_0^\infty sA_2(s)\cos(sx)\exp(-\gamma_2 y)ds \tag{23}$$

where $\gamma_1^2 = s^2 - \dfrac{\omega^2}{c_1^2\overline{\alpha}_0(s)}$, $\gamma_2^2 = s^2 - \dfrac{\omega^2}{c_2^2\overline{\alpha}_0(s)}$, $c_1 = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$, $c_2 = \sqrt{\dfrac{\mu}{\rho}}$ (24)

Now, let the function $A(s)$ be defined such that

$$A_1(s) = -\frac{1}{2\gamma_1}[s^2+\gamma_2^2]\overline{\alpha}_0(s)A(s) \tag{25}$$

$$A_2(s) = s\overline{\alpha}_0(s)A(s) \tag{26}$$

With the aid of equations (5),(6),(22),(23),(25) and (26), the appropriate quantities in equations (8) and (9) may be found to yield

$$\int_0^\infty A(s)\cos(sx)ds = 0 \quad , \quad x\rangle l \tag{27}$$

$$\int_0^\infty \overline{\alpha}_0^2(s)f(s)A(s)\cos(sx)ds = \frac{\pi\tau_0}{2\mu} \quad , \quad 0\langle x \leq l \tag{28}$$

The equations (27) and (28) are the dual integral equations of this problem. In equation (28), $f(s)$

is given as follows:

$$f(s) = \frac{1}{2\gamma_1}\{[s^2 + \gamma_2^2]^2 - 4s^2\gamma_1\gamma_2\} \tag{29}$$

5. Solution of the Dual Integral Equation

The dual integral equations (27) and (28) can not be transformed into the second kind Fredholm integral equation [3,4]. The kernel of the second kind Fredholm integral equation as in the references [3,4] is divergent. Of course, the dual integral equations can be considered to be a single integral equation of the first kind with a discontinuous kernel [2]. It is well-known in the literature that integral equations of the first kind are generally ill-posed in sense of Hadamard, i.e. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. In this paper, Schmidt's method was used to overcome the difficulty. The only difference between the classical and non-local equations is in the introduction of the function $\overline{\alpha}_0(s)$. As discussed in [3,4], it can be taken

$$\alpha_0 = \chi_0 \exp(-(\beta/a)^2(x'-x)^2), \text{ with } \qquad \chi_0 = \frac{1}{\sqrt{\pi}}\beta/a \tag{30}$$

where β is a constant, a is the lattice parameter.

So it can be obtained: $\overline{\alpha}_0(s) = \exp(-\frac{(sa)^2}{(2\beta)^2})$ (31)

$\overline{\alpha}_0(s) = 1$ for the limit $a \to 0$, so that the equation (28) reverts to the well-known equation of the classical theory. Here Schmidt method [11] can be used to solve the dual integral equations (27) and (28). The displacement v was represented by the following series:

$$v = \sum_{n=1}^{\infty} a_n P_{2n-2}^{(\frac{1}{2},\frac{1}{2})}(\frac{x}{l})(1-\frac{x^2}{l^2})^{\frac{1}{2}}, \qquad \text{for } |x| \le l, y = 0 \tag{32}$$

$$v = 0 \qquad , \qquad \text{for } |x| > l, y = 0 \tag{33}$$

where a_n are unknown coefficients to be determined and $P_n^{(\frac{1}{2},\frac{1}{2})}(x)$ is a Jacobi polynomial [12]. The Fourier transformation of equation (32) is [13]

$$-\frac{\omega^2}{2c_2^2}A(s)=\bar{v}(s,0,t)=\sum_{n=1}^{\infty}a_n G_n \frac{1}{s}J_{2n-1}(ls) \text{ with } G_n=2\sqrt{\pi}(-1)^{n-1}\frac{\Gamma(2n-\frac{1}{2})}{(2n-2)!} \tag{34}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting equation (34) into equations (27) and (28), respectively, the equation (27) has been automatically satisfied, the equation (28) reduces to the form for $|x|\le l$

$$\sum_{n=1}^{\infty}a_n G_n \int_0^{\infty}\bar{\alpha}_0^2(s)f(s)\frac{1}{s}J_{2n-1}(ls)\cos(xs)ds=-\frac{\pi\tau_0\omega^2}{4\mu c_2^2} \tag{35}$$

For a large s, the integrands of the equation (35) almost all decrease exponentially. So the semi-infinite integral in equation (35) can be evaluated numerically by Filon's method [14]. Thus equation (28) can be solve for coefficients a_n by the Schmidt method [11]. For brevity, the equation (35) can be rewritten as

$$\sum_{n=1}^{\infty}a_n E_n(x)=U(x) \tag{36}$$

where $E_n(x)$ and $U(x)$ are known functions and coefficients a_n are unknown and to be determined. A set of functions $P_n(x)$ which satisfy the orthogonality condition

$$\int_0^l P_m(x)P_n(x)dx=N_n\delta_{mn}\ , \qquad N_n=\int_0^l P_n^2(x)dx \tag{37}$$

can be constructed from the function, $E_n(x)$, such that

$$P_n(x)=\sum_{i=1}^{n}\frac{M_{in}}{M_{nn}}E_n(x) \tag{38}$$

where M_{ij} is the cofactor of the element d_{ij} of D_n, which is defined as

$$D_n=\begin{bmatrix} d_{11},d_{12},d_{13},\dots,d_{1n}\\ d_{21},d_{22},d_{23},\dots,d_{2n}\\ d_{31},d_{32},d_{33},\dots,d_{3n}\\ \dots\dots\dots\dots\\ \dots\dots\dots\dots\\ \dots\dots\dots\dots\\ d_{n1},d_{n2},d_{n3},\dots,d_{nn}\end{bmatrix},\quad d_{ij}=\int_0^l E_i(x)E_j(x)dx \tag{39}$$

Using equations (36) and (38), it can be obtained

$$a_n=\sum_{j=n}^{\infty}q_j\frac{M_{nj}}{M_{jj}} \qquad \text{with} \qquad q_j=\frac{1}{N_j}\int_0^l U(x)P_j(x)dx \tag{40}$$

6. Numerical Calculations And Discussion

Coefficients a_n are known, so that entire stress field is obtainable. However, in fracture mechanics, it is of importance to determine stress τ_{yy} along the crack line. τ_{yy} at y=0 is given as follows:

$$\tau_{yy} = \frac{4\mu c_2^2}{\pi\omega^2}\sum_{n=1}^{\infty} a_n G_n \int_0^{\infty} \overline{\alpha}_0^2 f(s)\frac{1}{s}J_{2n-1}(ls)\cos(sx)ds \tag{41}$$

For a=0 at x=l we have the classical stress singularity. However, so long as $a \neq 0$, (41) given a finite stress all along $y = 0$. At $0 < x < l$, τ_{yy} / τ_0 is very close to unity, and for $x > l$, τ_{yy} / τ_0 possesses finite values diminishing from a maximum value at $x = l$ to zero at $x = \infty$. The dynamic stress is computed numerically for the Lame constants $\lambda = 98 \times 10^9 (N/m^2)$, $\mu = 77 \times 10^9 (N/m^2)$, $\rho = 7.7 \times 10^3 (Kg/m^3)$. The semi-infinite numerical integrals, which occur, are evaluated easily by Filon and Simpson's methods because of the rapid diminution of the integrands. From references [15,16], it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of infinite series to equation (35) are retained. The results are plotted in Fig.2~9.

The following observations are very significant:

(i) The maximum normal stress occurs at the crack tip, and it is finite.

(ii) The normal stress at the crack tip becomes infinite as the atomic distance $a \to 0$. This is the classical continuum limit of square root singularity.

(iii) For the a/β =constant, viz., the atomic distance does not change, the values of the dynamic stress concentrations (at the crack tip) becomes higher with the increase of the crack length. Note this fact, experiments indicate that materials with smaller cracks are more resistant to fracture than those with larger cracks.

(iv) The stresses increase with the frequency ω becoming larger.

(v) The significance of this result is that the fracture criteria are unified at both the macroscopic and microscopic scales.

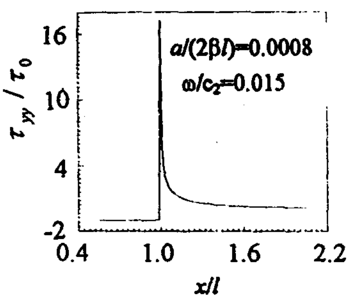

Fig.2. Normal stress

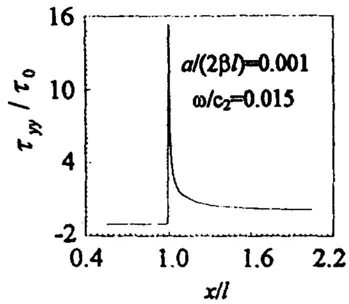

Fig.3. Normal stress

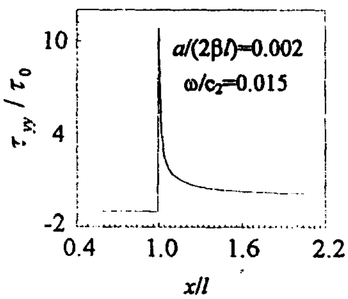

Fig.4. Normal stress

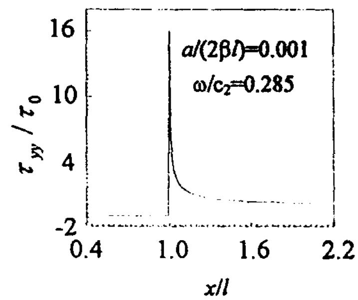

Fig.5. Normal stress

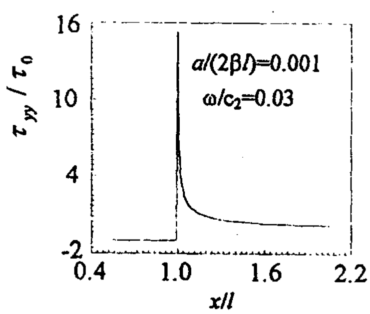

Fig.6. Normal stress

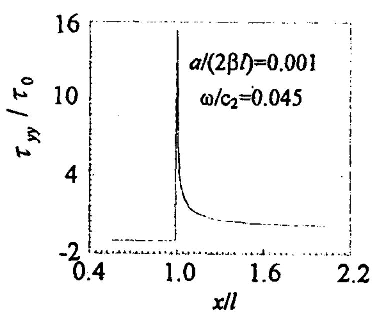

Fig.7. Normal stress

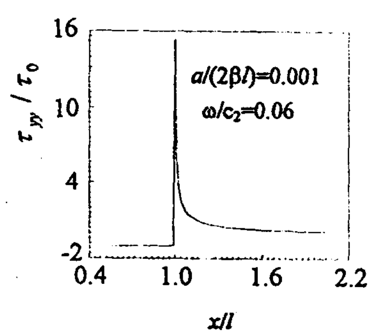

Fig.8. Normal stress

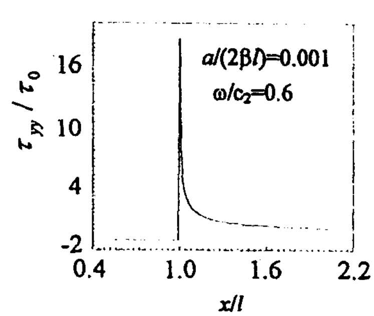

Fig.9. Normal stress

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