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TWO-DIMENSIONAL ELECTROELASTIC FUNDAMENTAL SOLUTIONS FOR GENERAL ANISOTROPIC PIEZOELECTRIC MEDIA*

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Abstract

Explicit fomulas for 2-D electroelastic fundamental solutions in general anisotropic piezoelectric media subjected to a line force and a line charge are obtained by using the plane wave decomposition method and a subsequent application of the residue calculus. "Anisotropic" means that any material symmetry restrictions are not assumed. "Two dimensional" includes not only in-plane problems but also anti-plane problems and problems in which in-plane and anti-plane deformations couple each other. As a special case, the solutions for transversely isotropic piezoelectric media are given.

Key words piezoelectric medium, plane wave decomposition method, electroelastic field, fundamental solution

I. Introduction

Due to their intrinsic coupling effect between mechanical and electrical fields, piezoelectric materials have been widely used in technology as transducers and sensors and, more recently, as actuators in smart structures.In order to optimize their microstructures and understand their fracture behaviours, several researchers have performed the analyses of piezoelectric materials containing an inclusion or a crack. Wang[1, 2] first analyzed the 3-D coupled electroelastic fields of a piezoelectric medium with an ellipsoidal inclusion and a flat elliptical crack by using the Green's function technique and Fourier transform. Using Stroh's formalism, Du et al.[3] obtained the electro-mechanical coupling fields of a 2-D anisotropic piezoelectric medium containing an elliptic inclusion, Pak[4] investigated dislocation and Griffith crack problems, and Suo et al.[5] studied in-body and interface crack problems of piezoelectric ceramics. Sosa[6] extended the Lekhnitskii's approach to investigate the plane problems in piezoelectric media with defects. Most works mentioned above are to solve the electroelastic fields in an unbounded piezoelectric medium subjected to uniform mechanical and electrical loading at infinity. In order to analyze a piezoelectric medium under complicated loading, it is very

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significant to search for the fundamental solutions for the coupled equations. Recently, Lee and Jiang[7] derived the fundamental solutions for the reduced 2-D anisotropic piezoelectric media using the Fourier transform; Meng and Du[8] gave the fundamental solutions for 2-D isotropic piezoelectric media empolying the same method as [7]. In the last two references, the complicated Fourier inversion must be performed to obtain the explicit expressions for the fundamental solutions. Because piezoelectric materials are naturally anisotropic and in-plane and anti-plane deformations often couple each other, the solutions given in Refs. [7] and [8] lack generality.

The main objective of this paper is to research the fundamental solutions for general anisotropic piezoeletric media subjected to a line force and a line charge. First of all, we obtained the integral representations of the fundamental solutions using the plane wave decomposition method, and then gave their explicit expressions by virtue of the residue calculus. The out of plane components of field variables are generally nonzero due to the anisotropy and electro-mechanical coupling effect of piezoelectric materials.Hence the present solutions are vaild not only for in plane problems but also for anti-plane problems and for the problems whose in-plane and anti-plane deformations couple each other. As a special case the analytical expressions of the fundamental solutions for transversely isotropic piezoelectric media are given.

II. Basic Equations

In a fixed rectangular coordinate system, x_i, the constitutive equations and gradient equations can be written as:

Constitutive equations

$$\sigma_{ij}=C_{ijmn}\varepsilon_{mn}-e_{nij}E_n,\quad D_i=e_{imn}\varepsilon_{mn}+\alpha_{in}E_n \tag{2.1}$$

where repeated indices imply summation, σ_{ij}, ε_{ij}, D_i and E_i are stress, strain, electric displacement and electric field respectively. C_{ijmn} are the elastic stiffnesses under constant electric field, e_{nij} are the piezoelectric stress constants, and α_{in} are the permittivities under constant strain field. They satisfy the symmetry relations

$$C_{ijmn}=C_{jimn}=C_{ijnm}=C_{mnij},\quad e_{nij}=e_{nji},\quad \alpha_{ij}=\alpha_{ji} \tag{2.2}$$

and positive definite property

$$C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}>0,\quad \alpha_{ij}E_iE_j>0 \tag{2.3}$$

Gradient equations

$$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2,\quad E_i=-\varphi_{,i} \tag{2.4}$$

where a comma stands for partial differentiation, u and φ are the elastic displacement and electric field respectively.

In the absence of body forces and free charges, the divergence equations are

$$\sigma_{ij,i}=0,\quad D_{i,i}=0 \tag{2.5}$$

Substituting Eqs. (2.1) and (2.4) into Eq. (2.5) yields

$$C_{ijmn}u_{m,ni}+e_{nij}\varphi_{,ni}=0,\quad e_{imn}u_{m,ni}-\alpha_{in}\varphi_{,ni}=0 \tag{2.6}$$

For convenience, the notation introduced by Barnett and Lothe[9] is employed to treat the elastic and electric variables on equal footing. Lower-case subscripts take on the range 1, 2 and 3, while upper-case subscripts take on the range 1, 2, 3 and 4. With this notation, the field variables can be expressed as

$$U_M=\begin{cases} u_m, & M=1,\ 2,\ 3 \\ \varphi, & M=4 \end{cases} \tag{2.7}$$

$$Z_{Mn}=\begin{cases} \varepsilon_{mn}, & M=1,\ 2,\ 3 \\ -E_n, & M=4 \end{cases} \tag{2.8}$$

$$\Sigma_{iJ}=\begin{cases} \sigma_{ij}, & J=1,\ 2,\ 3 \\ D_i, & J=4 \end{cases} \tag{2.9}$$

$$E_{iJMn}=\begin{cases} C_{ijmn}, & J,\ M=1,\ 2,\ 3 \\ e_{nij}, & J=1,\ 2,\ 3;\ M=4 \\ e_{imn}, & J=4;\ M=1,\ 2,\ 3 \\ -a_{in}, & J,\ M=4 \end{cases} \tag{2.10}$$

It is important to note that they are not tensors. Thus, one has to be careful when the coordinate system is changed.

According to the notation of Eqs. (2.7)~(2.10), the constitutive equations (2.1) and divergence equations (2.6) are written as

$$\Sigma_{iJ}=E_{iJMn}Z_{Mn} \tag{2.11}$$

$$E_{iJMn}U_{M,ni}=0 \tag{2.12}$$

III. Illustration of the Solution Method

The best way to describe the method of solution is to give a simple example. Thus, Let us consider the fundamental solution of Laplace's equation. It should satisfy

$$\nabla^2 G(\boldsymbol{x})+\delta(\boldsymbol{x})=0 \tag{3.1}$$

where ∇^2 denotes two-deimensional Laplacian operator.

The starting point of solving Eq. (3.1) is based on the use of the plane wave decomposition

$$\delta(\boldsymbol{x})=\frac{1}{4\pi^2}\nabla^2\oint_C \frac{1}{|\mathbf{s}|}\log|\mathbf{s}\cdot\boldsymbol{x}|ds \tag{3.2}$$

of the two-dimensional delta-function given in Ref. [10]. Where C is any closed curve enclosing the origin points $\mathbf{s}=0$ in $\mathbf{s}$ space and

$$ds=s_1 ds_2-s_2 ds_1 \tag{3.3}$$

Comparing Eq. (3.1) with Eq. (3.2) yields the integral representation of the function $G(\boldsymbol{x})$ as follows

$$G(\boldsymbol{x})=-\frac{1}{4\pi^2}\oint_C \frac{1}{|\mathbf{s}|^2}\log|\mathbf{s}\cdot\boldsymbol{x}|ds \tag{3.4}$$

In order to apply the residue calculus to Eq. (3.4), consider the closed contour shown in Fig. 1. It is easy to show that the contributions from C_2 and C_4 are zero as $|C_1|=|C_3|\to\infty$

and the contribution from C_3 equals the contribution from C_1. Thus, Eq. (3.4) reduces to

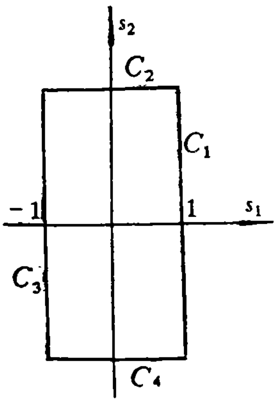

Fig. 1 Integral contour

$$G(\boldsymbol{x})=-\frac{1}{2\pi^2}\int_{C_1}\frac{1}{|\mathbf{s}|^2}\log|\mathbf{s}\cdot\boldsymbol{x}|ds$$
$$=-\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{1}{1+s_2^2}\log(x_1+s_2x_2)ds_2$$
$$=-\frac{1}{2\pi^2}\mathrm{Re}\int_{-\infty}^{\infty}\frac{1}{1+p^2}\log(x_1+px_2)dp \tag{3.5}$$

where Re denotes the real part. Evaluating Eq. (3.5) by the residue calculus yields the residue of the pole at $p=i$. The result is

$$G(\boldsymbol{x})=-\frac{1}{2\pi}\mathrm{Re}\ \log(x_1+ix_2)=\frac{1}{2\pi}\ \log\frac{1}{r}\qquad \left(r=|\boldsymbol{x}|=\sqrt{x_1^2+x_2^2}\right) \tag{3.6}$$

The above formula is the well-known fundamental solution of Laplace's equation.

IV. Fundamental Solutions for an Anisotropic Piezoelectric Medium

In this section, we will derive the 2-D fundamental solutions for general anisotropic piezoelectric media using the method in section III.

Consider an unbounded homogeneous anisotropic piezoelectric medium subjected to a line force and a line charge uniformly distributed over x_3-axis. Thus, the response electroelastic fields are dependent on x_1 and x_2 only. The fundamental solutions for piezoeloctric media are denoted by $G_{JM}(\boldsymbol{x})$, Their physical interpretations are: G_{mj} and G_{m4} represtent the elastic displacements along the $\boldsymbol{x}_m$ direction at $\boldsymbol{x}$ due to a unit line force along the $\boldsymbol{x}_j$ direction and a unit line charge at the O; G_{4j} and G_{44} represent the electric potential due to a unit line force along the x_j direction and a unit line charge at the origin O. The electroelastic fundamental solutions $G_{JM}(\boldsymbol{x})$ should satisfy the following system of partial differential equations:

$$\Gamma_{JM}(\partial_1,\ \partial_2)G_{MR}(\boldsymbol{x})+\delta_{JR}\delta(\boldsymbol{x})=0 \tag{4.1}$$

where

$$\Gamma_{JM}(\partial_1,\ \partial_2)=E_{\alpha JM\beta}\partial_\alpha\partial_\beta\qquad(\alpha,\ \beta=1,\ 2) \tag{4.2}$$

For any differentiable function $f(\mathbf{s}\cdot\boldsymbol{x})$, we observe that

$$\partial_\alpha f(\mathbf{s}\cdot\boldsymbol{x})=s_\alpha\dot{f}(\mathbf{s}\cdot\boldsymbol{x}) \tag{4.3}$$

where an overdot denotes the differentiation with respect to the argument.

By virtue of Eq. (4.3), it can be proven that

$$\Gamma_{JM}(\partial_1,\ \partial_2)\oint_C\Gamma_{MR}^{-1}(\mathbf{s})\log(\mathbf{s}\cdot\boldsymbol{x})ds$$
$$=\delta_{JR}\nabla^2\oint_C\frac{1}{|\mathbf{s}|^2}\log(\mathbf{s}\cdot\boldsymbol{x})ds \tag{4.4}$$

where

$$\Gamma_{JM}(s)=E_{\alpha JM\beta}s_\alpha s_\beta,\quad \Gamma_{JM}(s)\cdot\Gamma^{-1}_{MR}(s)=\delta_{JP} \tag{4.5}$$

In terms of Eqs. (2.2) and (2.3), it can be shown that $\Gamma_{JM}(s)$ is symmetric and non-singular. Therefore, the matrix $\Gamma^{-1}_{JM}(s)$ of $\Gamma_{JM}(s)$ exists. It follows from Eq. (4.5) that $\Gamma_{JM}(s)$ and $\Gamma^{-1}_{JM}(s)$ have the following properties, respectively

$$\Gamma_{JM}(\lambda s)=\lambda^2\Gamma_{JM}(s),\quad \Gamma^{-1}_{JM}(\lambda s)=\lambda^{-2}\Gamma^{-1}_{JM}(s) \tag{4.6}$$

It follows from Eq. (4.4) and Eq. (3.2) that the solutions $G_{RM}(\boldsymbol{x})$ which satisfy Eq. (4.1) are

$$G_{MR}(\boldsymbol{x})=-\frac{1}{4\pi^2}\oint_C \Gamma^{-1}_{MR}(s)\log|s\cdot\boldsymbol{x}|ds \tag{4.7}$$

Using Eq. (4.6) and the same procedure as section III, we reduce Eq. (4.7) to

$$\begin{aligned} G_{MR}(x)&=-\frac{1}{4\pi^2}\mathrm{Re}\int_{-\infty}^{\infty}\Gamma^{-1}_{MR}(1,\ p)\log(x_1+px_2)dp \\ &=-\frac{1}{4\pi^2}\mathrm{Re}\int_{-\infty}^{\infty}\frac{\Gamma^*_{MR}(p)}{D(p)}\log(x_1+px_2)dp \end{aligned} \tag{4.8}$$

where

$$\Gamma^*_{MR}(p)=\mathrm{adj}[\Gamma_{MR}(1,\ p)],\quad D(p)=\det[\Gamma_{MR}(1,\ p)] \tag{4.9}$$

We can see that $\Gamma^*_{MR}(p)$ and $D(p)$ are polynomial functions of p of order six and eight respectively. It can be shown that $D(p)$ does not have real roots due to the positive definiteness of the tensors C_{ijmn} and a_{ij}. Consequentely, there are four pair complex roots satisfying

$$D(p_m)=0 \qquad (m=1,\ 2,\ \cdots,\ 8) \tag{4.10}$$

They can be arranged as

$$p_{m+4}=\bar{p}_m,\quad \mathrm{Im}(p_m)>0 \qquad (m=1,\ 2,\ 3,\ 4) \tag{4.11}$$

where an overbar denotes the complex conjugate and Im stands for imaginary part. $D(p)$ can be expressed as:

$$D(p)=\sum_{i=0}^{8}a_ip^i=a_8\prod_{m=1}^{8}(p-p_m)(p-\bar{p}_m) \tag{4.12}$$

where a_i are the coefficients of the eighth polynomial function. Assuming that the roots of $D(p)$ are distinct and calculating the integral in (4.8) by means of the residue calculus, the explicit expression of $G_{MR}(\boldsymbol{x})$ is

$$G_{MR}(x)=\frac{1}{\pi}\mathrm{Im}\left\{\sum_{m=1}^{4}\frac{\Gamma^*_{MR}(p_m)}{\partial_pD(p_m)}\log(z_m)\right\} \tag{4.13}$$

where

$$z_m=x_1+p_mx_2 \tag{4.14}$$

For certain piezoelectric solids, such as transversely isotropic piezoelectric media, $D(p)$ has possibly multiple roots. For this case, we say the piezoelectric medium is degenerate. For the degenerate piezoelectric medium, Eq. (4.13) can be modified as follows

$$G_{MR}(x)=\frac{1}{\pi}\mathrm{Im}\left\{\sum_{m=1}^{r}\partial_{p}^{(t-1)}\left[\frac{\Gamma^{*}_{MR}(p_m)}{D_m(p_m)}\log(z_m)\right]\right\} \tag{4.15}$$

where t stands for the multiplicity of each root and r denotes the number of distinct roots. $D_m(p)$ is

$$D_m(p)=\eta_m D(p)/(p-p_m)^t \tag{4.16}$$

where

$$\eta_m=\begin{cases}1, & t=1,2\\ 1/2, & t=3\\ 1/6, & t=4\end{cases} \tag{4.17}$$

Eq. (4.15) is the explicit formula of the fundamental solutions for general anisotropic piezoelectric media. When the piezoelectric stress constants e_{nij} vanish, we can obtain the fundamental solutions for anisotropic elastic solids and dielectrics.

Note that in Eqs. (4.13) and (4.15) the following, notation is used.

$$\partial_p f(p_m)=[\partial f/\partial p]_{p=p_m} \tag{4.18}$$

V. A Special Case: Fundamental Solutions for Transversely Isotropic Piezoelectric Media

Many of piezoelectric materials which have widely been used in industry exhibit transversely isotropic piezoelectric behavior, such as piezoelectric ceramics. Assuming that x_1-x_2 plane is the isotropic plane and x_3-axis is parallel to the poling direction, the constitutive equations for these materials are:

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{Bmatrix}-\begin{bmatrix}0 & 0 & e_{31}\\ 0 & 0 & e_{31}\\ 0 & 0 & e_{33}\\ 0 & e_{16} & 0\\ e_{16} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}E_1\\ E_2\\ E_3\end{Bmatrix} \tag{5.1}$$

$$\begin{Bmatrix}D_1\\ D_2\\ D_3\end{Bmatrix}=\begin{bmatrix}0 & 0 & 0 & 0 & e_{15} & 0\\ 0 & 0 & 0 & e_{15} & 0 & 0\\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{13}\end{Bmatrix}+\begin{bmatrix}\alpha_{11} & 0 & 0\\ 0 & \alpha_{11} & 0\\ 0 & 0 & \alpha_{33}\end{bmatrix}\begin{Bmatrix}E_1\\ E_2\\ E_3\end{Bmatrix}$$

For the two-dimensional problems of transversely isotropic piezoelectric media, it follows from Eq. (4.5) that $\Gamma^{-1}(1,\ p)$ is

$$\Gamma^{-1}(1,\ p)=\begin{bmatrix}\Gamma_e^{-1}(1,\ p) & 0\\ 0 & \Gamma_p^{-1}(1,\ p)\end{bmatrix} \tag{5.2}$$

where

$$\Gamma_e^{-1}(1,\ p)=\frac{1}{C_{11}(C_{11}-C_{12})(p^2+1)^2}\begin{bmatrix}2C_{11}p^2+C_{11}-C_{12} & -(C_{11}+C_{12})p\\ -(C_{11}+C_{12})p & 2C_{11}+(C_{11}-C_{12})p^2\end{bmatrix} \tag{5.3}$$

$$\Gamma_p^{-1}(1,\ p)=\frac{1}{(e_{15}^2+\alpha_{11}C_{44})(1+p^2)}\begin{bmatrix}\alpha_{11} & e_{15}\\ e_{15} & -C_{44}\end{bmatrix} \tag{5.4}$$

where $\Gamma_e^{-1}(1,\ p)$ represents the matrix corresponding to in-plane deformation $(u_1,\ u_2)$, while $\Gamma_p^{-1}(1,\ p)$ denotes the matrix correspoinding to the coupled anti-plane deformation and electric field $(u_3,\ \varphi)$. After simple derivation, we obtain the nonzero components of the fundamental solutions as follows:

$$\begin{bmatrix}G_{11}(\boldsymbol{x}) & G_{12}(\boldsymbol{x})\\ G_{21}(\boldsymbol{x}) & G_{22}(\boldsymbol{x})\end{bmatrix}$$

$$=\frac{1}{8\pi C_{66}(1-\nu_{12})}\begin{bmatrix}(3-4\nu_{12})\log(1/r)-y^2/r^2 & xy/r^2\\ xy/r^2 & (3-4\nu_{12})\log(1/r)+y^2/r^2\end{bmatrix} \tag{5.5}$$

$$\begin{bmatrix}G_{33}(\boldsymbol{x}) & G_{34}(\boldsymbol{x})\\ G_{43}(\boldsymbol{x}) & G_{44}(\boldsymbol{x})\end{bmatrix}=\frac{1}{2\pi(1+k)}\begin{bmatrix}C_{44}^{-1} & ke_{15}^{-1}\\ ke_{15}^{-1} & -\alpha_{11}^{-1}\end{bmatrix}\log\frac{1}{r} \tag{5.6}$$

where ν_{12} is the Poisson's ratio, $k=e_{15}^2/(C_{44}\alpha_{11})$. Eq. (5.5) is identical to the classical results of the fundamental solutions for isotropic solids.

VI. Conclusions

Based on the plane wave decomposition method and the residue calculus, explicit formulas for the 2-D electoelastic fundamental solutions in general anisotropic piezoelectric media are obtained. The present method is able to deal with the "so-called degenerate materials" easily and avoids the complicated Fourier inversion.The procedure of the solution method is simple and clear. These solutions are essential to using the Boundary Element method to solve the coupled electroelastic field of piezoelectric media under general loading.

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