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ELECTRO-ELASTIC GREEN'S FUNCTIONS FOR A PIEZOELECTRIC HALF-SPACE AND THEIR APPLICATION*

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Abstract

In this paper, as is studied are the electro-elastic solutions for a piezoelectric half-space subjected to a line force, a line charge and a line dislocation, i. e., Green's functions on the basis of Stroh formalism and the concept of analytical continuation, explicit expressions for Green's functions are derived. As a direct application of the results obtained, an infinite piezoelectric solid containing a semi-infinite crack is examined. Attention if focused on the stress and electric displacement fields of a crack tip. The stress and electric displacement intensity factors are given explicitly.

Key words piezoelectric half-space, Green's function, semi-infinite crack, stress intensity factor, electric displacement intensity factor

I. Introduction

It is well-known that one of the most powerful tools in linear field theories is the Green's function. For elasticity, considerable research can be found in the literature. However, the Green's function for piezoelectricity is rather limited due to the anisotropy and electromechanical coupling effect of piezoelectric materials. Recent developments include: using the Fourier transformation techniques, Lee and Jiang[1] and Meng and Du[2] derived the Green's functions for the reduced 2—D transversely isotropic piezoelectric media and for the 2—D isotropic piezoelectric media, respectively; The authors[3] of the present paper presented the Green's functions for general anisotropic piezoelectric materials by empolying the plane wave decomposition method; Sosa and Castro[4] extended the state space method for elasticity to analyze the transversely isotropic piezoelectric half-plane, where a concentrated force and a point charge are applied at the boundary of the half-plane. Fan, Sze and Yang[5] studied the piezoelectric contact problem using the Stroh formalism. To the best of author's knowledge, no solutions for general anisotropic piezoelectric half-space under line forces, line charges and line dislocations have been reported.

In this paper, the simple explicit expressions of Green's functions for piezoelectric half-spaces are derived by using the Stroh formalism and the method of analytical continuation. As a direct application of the solutions obtained, the Green's functions for a piezoelectric

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medium containing a semi-infinite crack are obtained easily. Attention is focused on the analysis of electro-elastic fields near the crack tip, and the stress and electric displacement intensity factors are given explicitly.

II. Basic Equations of Piezoelectricity

In a fixed rectangular coordinate system (x_1, x_2, x_3), the field equations of linear piezoelectricity can be written as

Constitutive laws:

$$\sigma_{ij}=C_{ijmn}\gamma_{mn}-e_{nij}E_n,\quad D_i=e_{imn}\gamma_{mn}+\varepsilon_{in}E_n \tag{2.1}$$

where repeated indices mean summation. σ_{ij}, γ_{ij}, D_i and E_i are stress, strain, electric displacement (or electric induction) and electric field, respectively. C_{ijmn} are the elastic moduli measured at a constant electric field, e_{nij} are the piezoelectric constants, ε_{in} are the dielectric constants measured at a constant strain field.

Deformation relations

$$\gamma_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad E_i=-\varphi_{,i} \tag{2.2}$$

where a comma denotes partial differentiation, u_i and φ are the elastic displacement and electric potential, respectively.

Equilibrium equations

$$\sigma_{ij,i}=0,\qquad D_{i,i}=0 \tag{2.3}$$

in which body forces and free charges are neglected.

Due to the similarities between elastic variables and electric variables, it is convenient to treat them on equal footing in the solution of piezoelectric boundary value problems. So, the following notation is introduced[6]

$$U_M=\begin{cases} u_m & (M=1,2,3) \\ \varphi & (M=4) \end{cases} \tag{2.4}$$

$$Z_{Mn}=\begin{cases} \gamma_{mn} & (M=1,2,3) \\ -E_n & (M=4) \end{cases} \tag{2.5}$$

$$\Sigma_{iJ}=\begin{cases} \sigma_{ij} & (J=1,2,3) \\ D_i & (J=4) \end{cases} \tag{2.6}$$

$$E_{iJMn}=\begin{cases} C_{ijmn} & (J,M=1,2,3) \\ e_{nij} & (M=4;\ J=1,2,3) \\ e_{imn} & (J=4;\ M=1,2,3) \\ -\varepsilon_{in} & (J,M=4) \end{cases} \tag{2.7}$$

where lower-case subscripts take on the range 1, 2 and 3, while upper-case subscripts take on the range 1, 2, 3 and 4. It should be pointed out that they are not tensors. Thus, one has to be careful when the coordinate system is changed.

In terms of (2.4) to (2.7), the constitutive laws (2.1) and equilibrium equations (2.3) can be expressed as:

$$\Sigma_{iJ}=E_{iJMn}Z_{Mn}=E_{iJMn}U_{M,n} \tag{2.8}$$

$$E_{iJMn}U_{M,ni}=0 \tag{2.9}$$

III. Stroh Formalism for Piezoelectricity

For the two-dimensional piezoelectric problems dependent only on x_1 and x_2, the general solutions for the generalized displacement $U^T=[u_1,u_2,u_3,\varphi]$ and the generalized stress function $\Phi^T=[\Phi_1,\Phi_2,\Phi_3,\Phi_4]$ can be written as[7].

$$\left.\begin{aligned} U&=\mathrm{Re}\sum_{\alpha=1}^{4}\{a_\alpha f_\alpha(z_\alpha)+\bar{a}_\alpha \overline{f(z_\alpha)}\}=2\mathrm{Re}[Af(z)] \\ \Phi&=\mathrm{Re}\sum_{\alpha=1}^{4}\{b_\alpha f_\alpha(z_\alpha)+\bar{b}_\alpha \overline{f(z_\alpha)}\}=2\mathrm{Re}[Bf(z)] \end{aligned}\right\} \tag{3.1}$$

where Re stands for the real part and the overbar denotes the complex conjugate. The superscript T represents the transpose. $f^T(z)=[f_1(z_1),\ f_2(z_2),\ f_3(z_3),\ f_4(z_4)]$ is an arbitrary function vector of the generalized complex variables $z_\alpha=x_1+p_\alpha x_2$, the choice of which depends on the boundary conditions provided by the given problems. The complex constants p_α and the two 4×4 complex matrices A and B are functions of the material constants E_{iJMn}.

The stress and electric displacement fields are related to Φ by

$$\left.\begin{aligned} \Sigma_1&=[\sigma_{11},\sigma_{12},\sigma_{13},D_1]^T=-\frac{\partial\Phi}{\partial x_2}=-\Phi_{,2} \\ \Sigma_2&=[\sigma_{21},\sigma_{22},\sigma_{23},D_2]^T=\frac{\partial\Phi}{\partial x_1}=\Phi_{,1} \end{aligned}\right\} \tag{3.2}$$

The matrices A and B satisfy the following orthogonality relations

$$\begin{bmatrix} A^T & B^T \\ \bar{A}^T & \bar{B}^T \end{bmatrix}\begin{bmatrix} B & \bar{B} \\ A & \bar{A} \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3.3}$$

and the closure relations

$$\left.\begin{aligned} AB^T+\overline{AB}^T&=I=BA^T+\overline{BA}^T \\ AA^T+\overline{AA}^T&=0=BB^T+\overline{BB}^T \end{aligned}\right\} \tag{3.4}$$

where I is the unit matrix. Equation (3.4) implies

$$AA^T=-iH/2,\quad BB^T=iL/2,\quad AB^T=(I-iS)/2 \tag{3.5}$$

where $i=\sqrt{-1}$. H, L and S are real. H and L are symmetric positive-definite. They can be computed directly from the electro-elastic constants E_{iJMn} by

$$\left.\begin{aligned} S&=\frac{1}{\pi}\int_0^\pi N_1(\theta)d\theta \\ H&=\frac{1}{\pi}\int_0^\pi N_2(\theta)d\theta \\ L&=-\frac{1}{\pi}\int_0^\pi N_3(\theta)d\theta \end{aligned}\right\} \tag{3.6}$$

where

$$\left.\begin{array}{l} N_1(\theta)=-T^{-1}(\theta)R^T(\theta),\quad N_2(\theta)=T^{-1}(\theta) \\ N_3(\theta)=R(\theta)T^{-1}(\theta)R^T(\theta)-Q(\theta) \end{array}\right\} \tag{3.7}$$

$$Q_{JM}(\theta)=E_{iJMn}\eta_i\eta_n,\quad R_{JM}(\theta)=E_{iJMn}\eta_i\xi_n,\quad T_{JM}(\theta)=E_{iJMn}\xi_i\xi_n \tag{3.8}$$

$$\eta^T=[\cos\theta,\ \sin\theta,\ 0],\quad \xi^T=[-\sin\theta,\ \cos\theta,\ 0] \tag{3.9}$$

IV. The Green's Functions for a Piezoelectric Half-Space

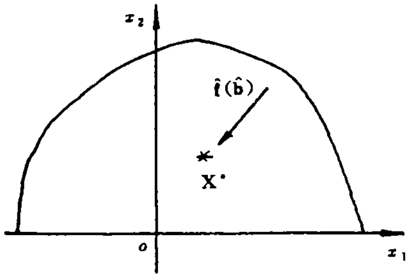

Fig. 1 A piezoelectric half-space subjected a line force, a line charge and a line dislocation

Consider a piezoelectric half-space $x_2\geqslant 0$, $-\infty<x_1<\infty$. The surface $x_2=0$ is traction-induction free. A line force, a line charge $\hat{f}^T=[f_1,f_2,f_3,\lambda]$ and a line dislocation with the generalized Burgers vector $\hat{b}^T=[b_1,b_2,b_3,\Delta\varphi]$ apply at the point $X^*=(x_1^*,x_2^*)$, where λ and $\Delta\varphi$ represent a line charge and an electric potential jump across the slip plane[8], respectively. The boundary conditions are

$$\left.\begin{array}{ll} \Phi=Bf(z)+\overline{BF(z)} & (x_2=0) \\ \left.\begin{array}{l}\oint_C d\Phi=\hat{f} \\ \oint_C \cdot dU=\hat{b}\end{array}\right\} & \text{(For any closed curve } C \text{ enclosing the point } X^*) \\ \Sigma_{ij}\to 0 & (|X|\to\infty) \end{array}\right\} \tag{4.1a~c}$$

To determine the solutions which satisfy the above conditions, assume that arbitrary functions $f_a(z_a)$ in the general solutions (3.1) are

$$f_a(z_a)=q_{a0}\ln(z_a-z_a^*)+g_a(z_a) \qquad (a=1,2,3,4,) \tag{4.2}$$

where $z_a^*=x_1^*+p_a x_2^*$; $q_{a0}\ln(z_a-\bar{z}_a^*)$ represent the singular solutions for an infinite homogenerous medium under $\hat{f}$ and $\hat{b}$, where q_{a0} are the unknown complex constants. $g_a(z_a)$ represent the perturbed solutions due to the boundary of a half-plane. Substitution of (4.2) into $(4.1)_1$ yields:

$$B_{ik}[q_{k0}\ln(x_1-z_k^*)+g_k(x_1)]+\overline{B_{ik}}[\bar{q}_{k0}\ln(x_1-\bar{z}_k^*)+\bar{g}_k(x_1)]=0 \tag{4.3}$$

Rearrange the above, one gets

$$\bar{B}_{ik}\bar{q}_{k0}\ln(x_1-\bar{z}_k^*)+B_{ik}g_k(x_1)=-B_{ik}q_{k0}\ln(x_1-z_k^*)-\bar{B}_{ik}\bar{g}_k(x_1) \tag{4.4}$$

in which the functions at the left-hand side are holomorphic in the upper half-plane, whereas those on the right-hand side are holomorphic in the lower half-plane. By the method of analytic continuation, we may introduce the funtion $F(z)$ which is holomorphic in the entire plane, i. e.

$$F(z)=\begin{cases} B_{ik}g_k(z)+\bar{B}_{ik}\bar{q}_{k0}\ln(z-\bar{z}_k^*) & (z\in S^+) \\ -\bar{B}_{ik}\bar{g}_k(z)-B_{ik}q_{k0}\ln(z-z_k^*) & (z\in S^-) \end{cases} \tag{4.5}$$

By Liouville's theorem, we hav

$$B_{ik}g_k(z)+\bar{B}_{ik}\bar{q}_{k0}\ln(z-\bar{z}_k^*)=0 \tag{4.6}$$

Solving the above equations and taking $z=z_a$ ($a=1, 2, 3, 4$), one finds

$$g_a(z_a)=-B_{ai}^{-1}\bar{B}_{ik}\bar{q}_{k0}\ln(z_a-\bar{z}_k^*) \tag{4.7}$$

and in a compact form

$$g(z)=-\sum_{k=1}^{4}\langle\ln(z_a-\bar{z}_k^*)\rangle B^{-1}\bar{B}I_k\bar{q}_0 \tag{4.8}$$

where

$$g^T(z)=[g_1(z_1),g_2(z_2),g_3(z_3),g_4(z_4)] \text{ and } q_0^T=[q_{10},q_{20},q_{30},q_{40}]$$

$$I_1=\mathrm{diag}[1,0,0,0],\ I_2=\mathrm{diag}[0,1,0,0],$$

$$I_3=\mathrm{diag}[0,0,1,0],\ I_4=\mathrm{diag}[0,0,0,1] \tag{4.9}$$

$$\langle\ln(z_a-\bar{z}_k^*)\rangle=\mathrm{diag}[\ln(z_1-\bar{z}_k^*),\ln(z_2-\bar{z}_k^*),\ln(z_3-\bar{z}_k^*),\ln(z_4-\bar{z}_k^*)] \tag{4.10}$$

To determine the unknown constant vector q_0, substituting (4.8) and (4.2) into (3.1) and using the conditions (4.1b) yields

$$\begin{pmatrix} A & \bar{A} \\ B & \bar{B} \end{pmatrix}\begin{pmatrix} q_0 \\ -\bar{q}_0 \end{pmatrix}=\frac{1}{2\pi i}\begin{pmatrix} \hat{b} \\ \hat{f} \end{pmatrix} \tag{4.11}$$

By the orthogonality relations (3.3), q_0 is

$$q_0=(A^T\hat{f}+B^T\hat{b})/2\pi i=h/2\pi i \tag{4.12}$$

In terms of (3.1), (4.2), (4.8) and (4.12), the Green's functions for a piezoelectric half-space can be written explicitly as

$$\left.\begin{aligned} U&=\frac{1}{\pi}\mathrm{Im}\left\{A\langle\ln(z_a-z_a^*)\rangle h+\sum_{k=1}^{4}A\langle\ln(z_a-\bar{z}_k^*)\rangle B^{-1}\bar{B}I_k\bar{h}\right\} \\ \Phi&=\frac{1}{\pi}\mathrm{Im}\left\{B\langle\ln(z_a-z_a^*)\rangle h+\sum_{k=1}^{4}B\langle\ln(z_a-\bar{z}_k^*)\rangle B^{-1}\bar{B}I_k\bar{h}\right\} \end{aligned}\right\} \tag{4.13}$$

where Im denotes the imaginary part.

From (3.2), the stress and electric displacement fields are

$$\left.\begin{aligned} \Sigma_1&=-\frac{1}{\pi}\mathrm{Im}\left\{B\left\langle\frac{p_a}{z_a-z_a^*}\right\rangle h+\sum_{k=1}^{4}B\left\langle\frac{p_a}{z_a-\bar{z}_a^*}\right\rangle B^{-1}\bar{B}I_k\bar{h}\right\} \\ \Sigma_2&=\frac{1}{\pi}\mathrm{Im}\left\{B\left\langle\frac{1}{z_a-z_a^*}\right\rangle h+\sum_{k=1}^{4}B\left\langle\frac{1}{z_a-\bar{z}_a^*}\right\rangle B^{-1}\bar{B}I_k\bar{h}\right\} \end{aligned}\right\} \tag{4.14}$$

Equation (4.14) shows that the condition (4.1c) has been satisfied.

V. An Infinite Piezoelectric Medium Containing a Semi-Infinite Crack

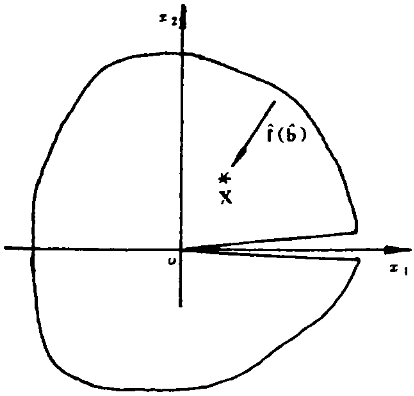

Fig. 2 An infinite piezoelectric solid with a semi-infinite crack

An infinite piezoelectric solid with a seim-infinite crack is shown in Fig. 2. The crack surface $x_2=0$, $0<x_1<\infty$ is traction-induction free. The same $\hat{f}$ and $\hat{b}$ as Fig. 1 apply at the point X^*. The solution of this problem can be obtained with the aid of the results in the above section.

Consider the conformal mapping function

$$z=\xi^2 \tag{5.1}$$

which maps the z plane with a crack into the upper half-plane in the ξ plane. So, replacing z_α, z_α^* and $\bar{z}_\alpha^*$ by $\sqrt{z_\alpha}$, $\sqrt{z_\alpha^*}$ and $\sqrt{\bar{z}_\alpha^*}$, we obtain

$$\left.\begin{aligned} U&=\frac{1}{\pi}\mathrm{Im}\left\{A\langle\ln(\sqrt{z_\alpha}-\sqrt{z_\alpha^*})\rangle h+\sum_{k=1}^{4}A\langle\ln(\sqrt{z_\alpha}-\sqrt{\bar{z}_k^*})\rangle B^{-1}\bar{B}I_k\bar{h}\right\}\\ \Phi&=\frac{1}{\pi}\mathrm{Im}\left\{B\langle\ln(\sqrt{z_\alpha}-\sqrt{z_\alpha^*})\rangle h+\sum_{k=1}^{4}B\langle\ln(\sqrt{z_\alpha}-\sqrt{\bar{z}_k^*})\rangle B^{-1}\bar{B}I_k\bar{h}\right\}\end{aligned}\right\} \tag{5.2}$$

By (3, 2), the stress and electric displacement fields in a piezoelectric medium can be expressed as

$$\Sigma_1=-\frac{1}{2\pi}\mathrm{Im}\left\{B\left\langle\frac{p_\alpha}{\sqrt{z_\alpha}(\sqrt{z_\alpha}-\sqrt{z_\alpha^*})}\right\rangle h+\sum_{k=1}^{4}B\left\langle\frac{p_\alpha}{\sqrt{z_\alpha}(\sqrt{z_\alpha}-\sqrt{\bar{z}_k^*})}\right\rangle B^{-1}\bar{B}I_k\bar{h}\right\}$$

$$\Sigma_2=\frac{1}{2\pi}\mathrm{Im}\left\{B\left\langle\frac{1}{\sqrt{z_\alpha}(\sqrt{z_\alpha}-\sqrt{z_\alpha^*})}\right\rangle h+\sum_{k=1}^{4}B\left\langle\frac{1}{\sqrt{z_\alpha}(\sqrt{z_\alpha}-\sqrt{\bar{z}_\alpha^*})}\right\rangle B^{-1}\bar{B}I_k\bar{h}\right\} \tag{5.3}$$

The above equations show that the electro-elastic fields at a crack tip are singular. The amplitudes of the singular fields are characterized by the stress and electric displacement intensity factors $K=[K_{\mathrm{II}},K_{\mathrm{I}},K_{\mathrm{III}},K_D]^T$, where K_D represents the electric displacement intensity factor. Similar to elasticity, K are given by

$$\begin{aligned} K&=[K_{\mathrm{II}},K_{\mathrm{I}},K_{\mathrm{III}},K_D]^T\\ &=\lim_{\substack{x_2\to 0\\ x_1\to 0}}\sqrt{-2\pi x_1}\,\Sigma_2\\ &=\frac{1}{\sqrt{2\pi}}\left\{\mathrm{Re}\left\{2B\left\langle\frac{1}{\sqrt{z_\alpha^*}}\right\rangle A^T\right\}\hat{f}+\frac{1}{\sqrt{2\pi}}\mathrm{Re}\left\{2B\left\langle\frac{1}{\sqrt{z_\alpha^*}}\right\rangle B^T\right\}\hat{b}\right. \end{aligned} \tag{5.4}$$

If $\hat{b}$ is absent and $\hat{f}$ apply on the crack surface $z_\alpha^*=x_0$, the above expression reduces to

$$K=\frac{1}{\sqrt{2\pi x_0}}\hat{f} \tag{5.5}$$

Equation (5.5) indicates that for some special cases the stress and electric displacement intensity factors are independent of the material properties.

VI. Conclusions

Based on the Stroh formalism and the method of analytical continuation, the simple explicit formula for the Green's functions in a piezoelectric half-space and an unbounded piezoelectric medium containing a semi-infinite crack are obtained. Emphasis is placed on the analysis of the electroelastic fields at the crack tip. The force-charge solutions of the present paper can be used as the fundamental solutions for the boundary element method to analyze the complicated electromechanical interaction problems, while dislocation solutions can be employed to study the interaction between a dislocation and boundaries and further compute the image forces acting on a dislocation.

The problem considered in this paper is that the surface $x_2=0$ or the crack surfaces is traction-induction free. For rigid boundary conditions, i. e. $U^T=[u_i, \varphi]=0$, letting $g(z)$ be replaced by

$$g(z)=-\sum_{k=1}^{4}\langle \ln(z_\alpha-\bar{z}_k^*)\rangle A^{-1}\bar{A}I_k\bar{q}_0 \tag{6.1}$$

one can obtains easily the Green's functions corresponding the rigid boundary conditions.

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