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# ELECTRO-ELASTIC FUNDAMENTAL SOLUTIONS OF ANISOTROPIC PIEZOELECTRIC MATERIALS WITH AN ELLIPTICAL HOLE*

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**ABSTRACT**: By using Stroh' complex formalism and Cauchy's integral method, the electro-elastic fundamental solutions of an infinite anisotropic piezoelectric solid containing an elliptic hole or a crack subjected to a line force and a line charge are presented in closed form. Particular attention is paid to analyzing the characteristics of the stress and electric displacement intensity factors. When a line force-charge acts on the crack surface, the real form expression of intensity factors is obtained. It is shown through a special example that the present work is correct.

**KEY WORDS**: piezoelectric material, fundamental solution, elliptic hole, intensity factor

## 1 INTRODUCTION

Piezoelectric materials exhibit coupling behavior between elastic and electric fields and are inherently anisotropic. They deform when subjected to an electric field and polarize when stressed. The coupling nature of the material has attracted wide applications in electro-mechanical and electronic devices such as electro-mechanical actuators, sensors and transducers. In addition, they play an important role in the emerging technologies of smart materials and structures. When subjected to mechanical and electrical loads, these piezoelectric materials can fail prematurely due to their brittleness and presence of defects or flaws such as inclusions, voids and cracks. Therefore, study on the electro-elastic interaction and fracture behaviors of piezoelectric materials has received some attention from the viewpoint of electro-mechanical coupling. Recently, using Stroh's formalism established by Stroh[1] and further elaborated by Ting[2] for two-dimensional anisotropic elasticity, Liang et al.[3] obtained the elastic and electric fields for a 2-D anisotropic piezoelectric medium containing an elliptic inclusion. Pak[4] gave the analytical solution to a transversely isotropic piezoelectric medium with a Griffith crack. Suo et al.[5] considered in-body and interface crack problems of piezoelectric ceramics. They gave the asymptotic solutions of the coupling fields near the crack tip and calculated the intensity factors and energy release rate. Sosa[6] extended Lekhnitskii's complex potential approach[7] to study the plane problem of piezoelectric media with defects, and then discussed the effect of electric field on the stress concentrated factor along the hole boundary. Wang[8] investigated the electro-elastic fields

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for a 3-D piezoelectric solid with a flat elliptical crack by using the Green's function and Fourier transformation techniques. He reached a conclusion that the electric displacement normal to the crack face influences the stress intensity factors. Wang and Zheng[9] analyzed the mechanical-electrical coupling behavior of a penny crack in piezoelectric ceramics under a lateral shearing force. All the studies mentioned above are restricted to the electro-elastic fields for an infinite piezoelectrics containing an inclusion or a crack under the far-field uniform mechanical and electrical loads.

In this paper, the electro-elastic fundamental solutions of an infinite piezoelectric solid containing an elliptic hole or a crack are derived in closed form by using Stroh's formalism and Cauchy's integral method. Particular emphasis is placed on analyzing the properties of stress and electric displacement intensity factors (SEIF). When a line force-charge applies on the crack surface, the real form expression for the SEIF's is given. It is shown through a special example that the present work is correct.

## 2 STROH'S FORMALISM FOR PIEZOELECTRIC MEDIA

In a fixed rectangular coordinate system $\chi_i (i = 1, 2, 3)$, the constitutive relations and equilibrium equations of linear piezoelectric media are

$$\Sigma_{iJ} = E_{iJMs} U_{M,s} \tag{1}$$

$$E_{iJMs} U_{M,si} = 0 \tag{2}$$

where body forces and free charges are neglected, a comma stands for differentiation, and repeated indices imply summation. Lowercase subscripts range from 1 to 3, while uppercase subscripts range from 1 to 4. $U_M$, $\Sigma_{iJ}$ and $E_{iJMs}$ are[11]

$$U_M = \begin{cases} u_m & M = 1,2,3 \\ \varphi & M = 4 \end{cases} \tag{3}$$

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij} & J = 1,2,3 \\ D_i & J = 4 \end{cases} \tag{4}$$

$$E_{iJMs} = \begin{cases} C_{ijms} & J, M = 1,2,3 \\ e_{sij} & J = 1,2,3 \qquad M = 4 \\ e_{ims} & J = 4 \qquad M = 1,2,3 \\ -\varepsilon_{is} & J, M = 4 \end{cases} \tag{5}$$

where $U_M$ and $\varphi$ are the elastic displacement and electric potential, respectively. $C_{ijms}, e_{sij}$ and $\varepsilon_{is}$ are the elastic constants, piezoelectric stress constants and dielectric constants, respectively. These material constants satisfy the symmetric relations

$$C_{ijms} = C_{jims} = C_{ijsm} = C_{msij} \qquad e_{sij} = e_{sji} \qquad \varepsilon_{is} = \varepsilon_{si} \tag{6}$$

and the positive definite property

$$C_{ijms} u_{i,j} u_{m,s} > 0 \qquad \varepsilon_{is} E_i E_s > 0 \tag{7}$$

The electric field $\boldsymbol{E}$ is given by

$$E_i = -\varphi_{,i} \tag{8}$$

For two-dimensional piezoelectric problems dependent only $x_1$ and $x_2$, a general solution to Eq.(2) can be expressed as[3]

$$\boldsymbol{U} = \boldsymbol{a} f(z) \qquad z = x_1 + p x_2 \tag{9}$$

where $\boldsymbol{U} = \{u_1, u_2, u_3, \varphi\}^{\mathrm{T}}$ and $\boldsymbol{a} = \{a_1, a_2, a_3, a_4\}^{\mathrm{T}}$. Substituting (9) into (2) yields

$$[\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p^2 \boldsymbol{T}]\boldsymbol{a} = 0 \tag{10}$$

In the above, the superscript T denotes the transpose and $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{T}$ are $4 \times 4$ matrices whose components are

$$Q_{JM} = E_{1JM1} \qquad R_{JM} = E_{1JM2} \qquad T_{JM} = E_{2JM2} \tag{11}$$

The matrices $\boldsymbol{Q}$ and $\boldsymbol{T}$ are symmetric, and can be shown to be nonsingular.

For non-trivial solution of $\boldsymbol{a}$, we must have

$$\det[\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p^2 \boldsymbol{T}] = 0 \tag{12}$$

This is an eigenvalue problem. As in the anisotropic elasticity formulation, it can be proved that the eigenvalue $p$ cannot be purely real by virtue of Eq.(7). By letting

$$p_{\alpha+4} = \bar{p}_\alpha \quad \mathrm{Im}(p_\alpha) > 0 \qquad \alpha = 1, 2, 3, 4 \tag{13}$$

where an overbar stands for the complex conjugate and Im denotes the imaginary part, the associated eigenvectors are

$$\boldsymbol{a}_{\alpha+4} = \bar{\boldsymbol{a}}_\alpha \tag{14}$$

With the above analysis, the general solution can be written as a linear combinations of the eight eigenvectors

$$\boldsymbol{U} = \sum_{\alpha=1}^{4} \left\{ \boldsymbol{a}_\alpha f_\alpha(z_\alpha) + \bar{\boldsymbol{a}}_\alpha \overline{f_\alpha(z_\alpha)} \right\} = 2\mathrm{Re} \sum_{\alpha=1}^{4} \boldsymbol{a}_\alpha f_\alpha(z_\alpha) \qquad z_\alpha = x_1 + p_\alpha x_2 \tag{15}$$

where Re denotes real part.

The stress and electric displacement fields obtained by inserting (15) into (1) can be expressed in terms of the generalized stress function vector $\boldsymbol{\Phi}$ as

$$\left.\begin{aligned} \boldsymbol{\Sigma}_1 &= \{\sigma_{11}, \sigma_{12}, \sigma_{13}, D_1\}^{\mathrm{T}} = -\boldsymbol{\Phi}_{,2} \\ \boldsymbol{\Sigma}_2 &= \{\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2\}^{\mathrm{T}} = \boldsymbol{\Phi}_{,1} \end{aligned}\right\} \tag{16}$$

where

$$\boldsymbol{\Phi} = \{\phi_1, \phi_2, \phi_3, \phi_4\}^{\mathrm{T}} = 2\mathrm{Re} \sum_{\alpha=1}^{4} \boldsymbol{b}_\alpha f(z_\alpha) \tag{17}$$

$$\boldsymbol{b} = (\boldsymbol{R}^{\mathrm{T}} + p\boldsymbol{T}) = -\frac{1}{p}(\boldsymbol{Q} + p\boldsymbol{R})\boldsymbol{a} \tag{18}$$

The two equations in (18) can be converted to a standard eigenvalue problem

$$\boldsymbol{N}\boldsymbol{y} = p\boldsymbol{y} \tag{19}$$

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 \\ \boldsymbol{N}_3 & \boldsymbol{N}_1^{\mathrm{T}} \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} \tag{20}$$

$$\boldsymbol{N}_1 = -\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}} \quad \boldsymbol{N}_2 = \boldsymbol{T}^{-1} \quad \boldsymbol{N}_3 = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}} - \boldsymbol{Q} \tag{21}$$

The $4 \times 4$ matrices $\boldsymbol{N}_2$ and $\boldsymbol{N}_3$ are symmetric due to the symmetric property of $\boldsymbol{Q}$ and $\boldsymbol{T}$. The eigenvalues of Eq.(19) can be obtained by solving

$$\det(\boldsymbol{N} - p\boldsymbol{I}) = 0 \tag{22}$$

in which $\boldsymbol{I}$ is the $4 \times 4$ unit matrix.

Finally, the general solution can be written in matrix notation as

$$\boldsymbol{U} = 2\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(\boldsymbol{z})\} = \boldsymbol{A}\boldsymbol{f}(\boldsymbol{z}) + \bar{\boldsymbol{A}}\overline{\boldsymbol{f}(\boldsymbol{z})} \tag{23}$$

$$\boldsymbol{\Phi} = 2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(\boldsymbol{z})\} = \boldsymbol{B}\boldsymbol{f}(\boldsymbol{z}) + \bar{\boldsymbol{B}}\overline{\boldsymbol{f}(\boldsymbol{z})} \tag{24}$$

The $4 \times 4$ matrices $\boldsymbol{A}, \boldsymbol{B}$ and the function vector $\boldsymbol{f}(\boldsymbol{z})$ are defined by

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4] \quad \boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4]$$
$$\boldsymbol{f}(\boldsymbol{z}) = \{f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)\}^{\mathrm{T}}$$

We note that for the piezoelectric boundary value problem, one has to determine unknown function vector $\boldsymbol{f}(\boldsymbol{z})$ according to the boundary conditions.

For further reference, some useful relations are listed next[3])

$$\begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{B}^{\mathrm{T}} \\ \bar{\boldsymbol{A}}^{\mathrm{T}} & \bar{\boldsymbol{B}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{B} & \bar{\boldsymbol{B}} \\ \boldsymbol{A} & \bar{\boldsymbol{A}} \end{bmatrix} = \boldsymbol{I} \tag{25}$$

and

$$\boldsymbol{S} = \mathrm{i}(2\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} - 1) \quad \boldsymbol{H} = 2\mathrm{i}\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} \quad \boldsymbol{L} = -2\mathrm{i}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}} \tag{26}$$

where $\mathrm{i} = \sqrt{-1}$. The matrices $\boldsymbol{S}, \boldsymbol{H}$ and $\boldsymbol{L}$ are real. The $\boldsymbol{H}$ and $\boldsymbol{L}$ are symmetric and nonsingular.

To avoid determining the eigenvalues $p$ and eigenvectors $\boldsymbol{y}$, the matrices $\boldsymbol{S}, \boldsymbol{H}$ and $\boldsymbol{L}$ can be computed directly from the material constants $E_{iJMs}$ by

$$\boldsymbol{S} = \frac{1}{\pi}\int_0^{\pi} \boldsymbol{N}_1(\theta)\mathrm{d}\theta \quad \boldsymbol{H} = \frac{1}{\pi}\int_0^{\pi} \boldsymbol{N}_2(\theta)\mathrm{d}\theta \quad \boldsymbol{L} = -\frac{1}{\pi}\int_0^{\pi} \boldsymbol{N}_3(\theta)\mathrm{d}\theta \tag{27}$$

where

$$\boldsymbol{N}_1 = -\boldsymbol{T}^{-1}(\theta)\boldsymbol{R}^{\mathrm{T}}(\theta) \quad \boldsymbol{N}_2(\theta) = \boldsymbol{T}^{-1}(\theta) \quad \boldsymbol{N}_3(\theta) = \boldsymbol{R}(\theta)\boldsymbol{T}^{-1}(\theta)\boldsymbol{R}^{\mathrm{T}}(\theta) - \boldsymbol{Q}(\theta) \tag{28}$$

$$\left.\begin{aligned} &Q_{JM}(\theta) = E_{iJMs}n_i(\theta)n_s(\theta) \qquad R_{JM}(\theta) = E_{iJMs}n_i(\theta)m_s(\theta) \\ &T_{JM}(\theta) = E_{iJMs}m_i(\theta)m_s(\theta) \end{aligned}\right\} \tag{29}$$

$$\boldsymbol{n} = \{\cos\theta, \sin\theta, 0\}^{\mathrm{T}} \qquad \boldsymbol{m} = \{-\sin\theta, \cos\theta, 0\}^{\mathrm{T}} \tag{30}$$

## 3 FUNDAMENTAL SOLUTION FOR ELLIPTIC HOLE PROBLEM

Consider an infinite anisotropic piezoelectric solid containing an elliptic hole with major axis $2a$ and minor $2b$ as shown in Fig.1. Assume that the hole is traction-charge free and a line force-charge $\boldsymbol{P} = \{P_1, P_2, P_3, P_4\}^{\rm T}$ applies at the point $z^* = x_1^* + {\rm i}x_2^*$, where $P_4$ represents a line charge. The boundary conditions of this problem are

$$\boldsymbol{\Phi}(z) = 0 \qquad z \in \Gamma \tag{31}$$

$$\oint_C {\rm d}\boldsymbol{\Phi} = \boldsymbol{P} \text{ for any closed curve } C \text{ enclosing the point } z^* \tag{32}$$

$$\Sigma_{iJ} \to 0 \qquad \text{when} \qquad |z| \to \infty \tag{33}$$

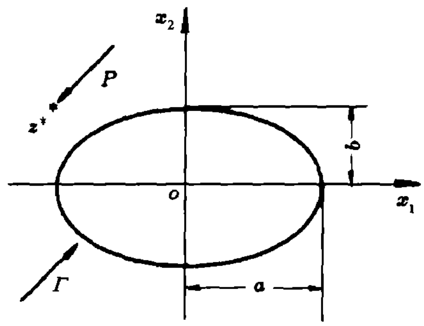


Fig.1 An infinite piezoelectric solid containing an elliptic hole

Assume that the functions $f_\alpha(z_\alpha)$ to be determined have the following form

$$f_\alpha(z_\alpha) = q_\alpha \log(z_\alpha - z_\alpha^*) + g_\alpha(z_\alpha) \qquad (\text{not sum on } \alpha) \tag{34}$$

where $q_\alpha \log(z_\alpha - z_\alpha^*)$ represent the singular solutions for an infinite medium under $\boldsymbol{P}$ and $q_\alpha$ are the complex constant to be determined, while $g_\alpha(z_\alpha)$ represent the perturbed solutions due to the presence of a hole and are holomorphic outside the elliptic boundary.

Following Ref.[7], we know that the conformal mapping functions

$$\zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - p_\alpha^2 b^2}}{a - {\rm i}p_\alpha b} \qquad \alpha = 1, 2, 3, 4 \tag{35}$$

will map the exterior of $\Gamma_\alpha$ in the $z_\alpha$ plane onto the exterior of a unit circle, $\sigma = {\rm e}^{{\rm i}\beta} (0 \le \beta \le 2\pi)$, in the $\zeta_\alpha$ plane and the four points on the contours of $\Gamma_\alpha$ into a single point on the contour of $\sigma$. After transformation, $f_\alpha(z_\alpha)$ can be written as

$$f_\alpha(z_\alpha) = q_\alpha \log(\zeta_\alpha - \zeta_\alpha^*) + g_\alpha(\zeta_\alpha) \tag{36}$$

where

$$\zeta_\alpha^* = \frac{z_\alpha^* + \sqrt{z_\alpha^{*2} - a^2 - p_\alpha^2 b^2}}{a - {\rm i}p_\alpha b} \qquad z_\alpha^* = x_1^* + p_\alpha x_2^* \tag{37}$$

Employing (24), (31) and (36), one obtains

$$B_{ik}[q_k \log(\sigma - \zeta_k^*) + g_k(\sigma)] + \bar{B}_{ik}[\bar{q}_k \log(\bar{\sigma} - \bar{\zeta}_k^*) + \overline{g_k(\sigma)}] \tag{38}$$

Note that $\bar{\sigma} = \sigma^{-1}$ on the unit circle, and $\overline{g_k(\sigma)} = \bar{g}_k(\sigma^{-1})$ is the boundary value of $\bar{g}_k(\zeta^{-1})$ which is holomorphic $|\zeta| < 1$. To solve (38) for the functions $g_\alpha(\zeta_\alpha)$, we multiply both sides by $\dfrac{1}{2\pi\mathrm{i}(\sigma - \zeta)}$ and integrate along the unit circle, where $\zeta$ is any point outside the unit circle. According to the Cauchy's formula for the infinite region[12], we obtain

$$B_{i\alpha} g_\alpha(\zeta) + \bar{B}_{ik}\bar{q}_k \log\left(\zeta^{-1} - \bar{\zeta}_k^*\right) = 0 \tag{39}$$

Solving the Eq.(39) and taking $\zeta = \zeta_\alpha$, we have

$$g_\alpha(\zeta_\alpha) = -B_{\alpha j}^{-1}\bar{B}_{jk}\bar{q}_k \log\left(\zeta_\alpha^{-1} - \bar{\zeta}_k^*\right) \tag{40}$$

where $B_{\alpha j}^{-1}B_{jk} = \delta_{\alpha k}$.

From (36) and (40), $\boldsymbol{f}(\boldsymbol{z})$ can be written in compact form as

$$\boldsymbol{f}(\boldsymbol{z}) = \langle\log(\zeta_\alpha - \zeta_\alpha^*)\rangle\boldsymbol{q} - \sum_{k=1}^{4}\langle\log(\zeta_\alpha^{-1} - \bar{\zeta}_k^*\rangle\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\bar{\boldsymbol{q}} \tag{41}$$

where

$$\boldsymbol{I}_1 = \mathrm{diag}[1,0,0,0] \qquad \boldsymbol{I}_2 = \mathrm{diag}[0,1,0,0]$$
$$\boldsymbol{I}_3 = \mathrm{diag}[0,0,1,0] \qquad \boldsymbol{I}_4 = \mathrm{diag}[0,0,0,1]$$
$$\langle F(\zeta_\alpha)\rangle = \mathrm{diag}[F(\zeta_1), F(\zeta_2), F(\zeta_3), F(\zeta_4)] \qquad \boldsymbol{q} = [q_1, q_2, q_3, q_4]^{\mathrm{T}}$$

Using (41), (23) and (24), $\boldsymbol{U}$ and $\boldsymbol{\Phi}$ are

$$\left.\begin{aligned} \boldsymbol{U} &= 2\mathrm{Re}\Big\{\boldsymbol{A}\langle\log(\zeta_\alpha - \zeta_\alpha^*)\rangle\boldsymbol{q}\Big\} - 2\sum_{k=1}^{4}\mathrm{Re}\Big\{\boldsymbol{A}\langle\log(\zeta_\alpha^{-1} - \bar{\zeta}_k^*)\rangle\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\bar{\boldsymbol{q}}\Big\} \\ \boldsymbol{\Phi} &= 2\mathrm{Re}\Big\{\boldsymbol{B}\langle\log(\zeta_\alpha - \zeta_\alpha^*)\rangle\boldsymbol{q}\Big\} - 2\sum_{k=1}^{4}\mathrm{Re}\Big\{\boldsymbol{B}\langle\log(\zeta_\alpha^{-1} - \bar{\zeta}_k^*)\rangle\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\bar{\boldsymbol{q}}\Big\} \end{aligned}\right\} \tag{42}$$

To determine $\boldsymbol{q}$, one uses Eq.(32) and the requirement of single-valued displacement, which lead to

$$\left.\begin{aligned} 2\mathrm{Re}(\mathrm{i}\boldsymbol{A}\boldsymbol{q}) &= 0 \\ 2\mathrm{Re}(\mathrm{i}\boldsymbol{B}\boldsymbol{q}) &= \frac{1}{2\pi}\boldsymbol{P} \end{aligned}\right\} \tag{43}$$

Employing relation (25) and solving Eq.(43), we obtain

$$\boldsymbol{q} = \frac{1}{2\pi\mathrm{i}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} \tag{44}$$

Substituting (44) into (42), the fundamental solutions can then be expressed as

$$\boldsymbol{U} = \frac{1}{\pi}\mathrm{Im}\Big\{\boldsymbol{A}\langle\log(\zeta_\alpha - \zeta_\alpha^*)\rangle\boldsymbol{A}^{\mathrm{T}}\Big\}\boldsymbol{P} + \frac{1}{\pi}\sum_{k=1}^{4}\mathrm{Im}\Big\{\boldsymbol{A}\langle\log(\zeta_\alpha^{-1}\bar{\zeta}_k^*)\rangle\boldsymbol{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\boldsymbol{A}^{\mathrm{T}}\Big\}\boldsymbol{P} \tag{45}$$

$$\boldsymbol{\Phi} = \frac{1}{\pi}\mathrm{Im}\Big\{\boldsymbol{B}\langle\log(\zeta_\alpha - \zeta_\alpha^*)\rangle\boldsymbol{A}^{\mathrm{T}}\Big\}\boldsymbol{P} + \frac{1}{\pi}\sum_{k=1}^{4}\mathrm{Im}\Big\{\mathrm{B}\langle\log(\zeta_\alpha^{-1} - \bar{\zeta}_k^*)\rangle\mathrm{B}^{-1}\bar{\boldsymbol{B}}\boldsymbol{I}_k\boldsymbol{A}^{\mathrm{T}}\Big\}\boldsymbol{P} \tag{46}$$

## 4 STRESS AND ELECTRIC DISPLACEMENT INTENSITY FACTORS

The fundamental solutions for an infinite piezoelectric medium with a crack of length $2a$ can be obtained easily by letting $b=0$ in (45) and (46). We are interested in the stress and electric displacement fields near the crack tips. Differentiating the generalized stress function $\Phi$ with respect to $x_1$ and considering that $x_2=0, x_1>a$, the stress and electric displacement $\boldsymbol{\Sigma}_2$ ahead of the crack tip along the $x_1$ axis are obtained as follows

$$\boldsymbol{\Sigma}_2=\{\sigma_{21},\sigma_{22},\sigma_{23},D_2\}^{\mathrm{T}}=$$
$$\frac{1}{\pi a}\Big(1+\frac{x_1}{\sqrt{x_1^2-a^2}}\Big)\mathrm{Im}\Big\{\boldsymbol{B}\Big\langle\frac{1}{\zeta-\zeta_\alpha^*}\Big\rangle\boldsymbol{A}^{\mathrm{T}}-\bar{\boldsymbol{B}}\Big\langle\frac{1}{\zeta-\zeta^2\bar{\zeta}_\alpha^*}\Big\rangle\bar{\boldsymbol{A}}^{\mathrm{T}}\Big\}\boldsymbol{P} \tag{47}$$

where

$$\zeta=\big(x_1+\sqrt{x_1^2-a^2}\big)/a \qquad \zeta_\alpha^*=\big(z_\alpha^*+\sqrt{z_\alpha^{*2}-a^2}\big)/a \tag{48}$$

Equation (47) indicates that the electro-elastic fields are singular at the crack tip. The amplitudes of the singular fields can be characterized by the stress and electric displacement intensity factors $\boldsymbol{K}^{\mathrm{T}}=\{K_2,K_1,K_3,K_D\}$, where $K_D$ denotes the electric displacement intensity factor.

Similar to anisotropic elasticity, $\boldsymbol{K}$ is given by

$$K=\{K_2,K_1,K_3,K_D\}^{\mathrm{T}}=$$
$$\lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\boldsymbol{\Sigma}_2=$$
$$\frac{1}{2\sqrt{\pi a}}\mathrm{Im}\Big\{\boldsymbol{B}\Big\langle1-\frac{z_\alpha^*+a}{\sqrt{z_\alpha^{*2}-a^2}}\Big\rangle\boldsymbol{A}^{\mathrm{T}}-\bar{\boldsymbol{B}}\Big\langle1-\frac{\bar{z}_\alpha^*+a}{\sqrt{\bar{z}_\alpha^{*2}-a^2}}\Big\rangle\bar{\boldsymbol{A}}^{\mathrm{T}}\Big\}\boldsymbol{P} \tag{49}$$

where we have used $\lim\limits_{x_1\to a}\zeta=1$ and the relation

$$\frac{1}{1-\zeta_\alpha^*}=\frac{1}{2}\Big(1-\frac{z_\alpha^*+a}{\sqrt{z_\alpha^{*2}-a^2}}\Big) \tag{50}$$

When $\boldsymbol{P}$ applies on the upper crack surface $x_1=c$, employing the jump property of the function $\big(z_\alpha^{*2}-a^2\big)^{-1/2}$ and $(26)_1$, the real form expression for $\boldsymbol{K}$ can be obtained as follows

$$\boldsymbol{K}=-\frac{1}{2\sqrt{\pi a}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{P}+\frac{1}{2\sqrt{\pi a}}\sqrt{\frac{a+c}{a-c}}\boldsymbol{P} \tag{51}$$

We find that the structure of the coupling intensity factors for piezoelectricity is identical with one of the stress intensity factors for anisotropy elasticity[10]. It is interesting to note that the first term on r.h.s. of (51) is only dependent on the material constants, while the second term is dependent on the location of a force-charge. When a pair of self-equilibrating forces and a pair of positive-negative charges act on the crack surfaces $x_1=c$, the stress and electric displacement intensity factors are

$$\boldsymbol{K}=\frac{1}{\sqrt{\pi a}}\sqrt{\frac{a+c}{a-c}}\boldsymbol{P} \tag{52}$$

Form (52), results given by Suo et al.[5] can be obtained by integrating.

As a special example, we will consider the coupling intensity factors for transversely isotropic piezoelectric materials. Assuming that the $x_1$-$x_2$ plane is the isotropic plane and the $x_3$ axis is parallel to the poling axis, the constitutive equations of these materials are

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} \tag{53}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{Bmatrix} + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} \tag{54}$$

Form $(27)_1$, integration yields

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1 & \\ & \boldsymbol{S}_2 \end{bmatrix} \quad \boldsymbol{S}_1 = \frac{C_{66}}{C_{12}+2C_{66}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \boldsymbol{S}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{55}$$

Substituting (55) into (51), we obtain

$$\left.\begin{aligned} K_1 &= \frac{C_{66}}{2(C_{12}+2C_{66})} \times \frac{P_1}{\sqrt{\pi a}} + \frac{1}{2}\sqrt{\frac{a+c}{a-c}} \times \frac{P_2}{\sqrt{\pi a}} \\ K_2 &= -\frac{C_{66}}{2(C_{12}+2C_{66})} \times \frac{P_2}{\sqrt{\pi a}} + \frac{1}{2}\sqrt{\frac{a+c}{a-c}} \times \frac{P_1}{\sqrt{\pi a}} \\ K_3 &= \frac{1}{2}\sqrt{\frac{a+c}{a-c}} \times \frac{P_3}{\sqrt{\pi a}} \\ K_D &= \frac{1}{2}\sqrt{\frac{a+c}{a-c}} \times \frac{P_4}{\sqrt{\pi a}} \end{aligned}\right\} \tag{56}$$

Letting $C_{12} = \frac{2\nu_{12}}{1-\nu_{12}} C_{66}$, where $\nu_{12}$ is the Poisson's ratio, $K_1$ and $K_2$ are identical with the classical ones for a crack in an isotropic material.

## 5 CONCLUDING REMARKS

Based on the Stroh formalism and the Cauchy integral method , the fundamental solutions of an anisotropic piezoelectric medium containing an elliptic hole and a crack are obtained in this paper. The novel features of the present work include: (1) the derivation is valid for general anisotropic piezoelectric materials which need not have any material symmetry restrictions; (2) the fundamental solutions have satisfied traction-charge free boundary conditions along an elliptic hole, and are suitable not only for plane problems but also for anti-plane problems. The results obtained in this paper can be used for analyzing the mechanical-electric behavior of piezoelectric materials with complicated configuration and geometry under arbitrary loads by combining the Boundary Element Method.

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