ACTA MECHANICA SINICA, Vol.10, No.3, August 1994 ISSN 0567-7718
Science Press, Beijing, China
Allerton Press, INC., New York, U.S.A.

GENERAL COUPLED SOLUTION OF ANISOTROPIC PIEZOELECTRIC MATERIALS WITH AN ELLIPTIC INCLUSION*

Du Shanyi (杜善义) Liang Jun (梁军) Han Jiecai (韩杰才) Wang Biao (王彪)
(*Harbin Institute of Technology, Harbin* 150001, *China*)

ABSTRACT: In this investigation, the Stroh formalism is used to develop a general solution for an infinite, anisotropic piezoelectric medium with an elliptic inclusion. The coupled elastic and electric fields both inside the inclusion and on the interface of the inclusion and matrix are given.

KEY WORDS: piezoelectric solids, inclusion, electroelastic field

I. INTRODUCTION

The coupled electroelastic behaviour of piezoelectric materials adds considerably to the difficulties in the design and analysis of mechanical behaviour of the materials. Further complications arise from the inherent anisotropic of the piezoelectric ceramics. Nevertheless, a reasonable amount of theoretical work has been directed towards the study of dislocation, crack and inclusion problems as well as interfacial behaviour in homogeneous piezoelectric solids. Sosa and Pak[1] developed a three-dimensional solution for isotropic piezoelectric ceramic with defects. Wang and Du[2,3] and Chen[4] analysed the inclusion and crack problems in piezoelectric matrix based on Green's function and Fourier transformation techniques. Extending Deeg's[5] rigorous analytical solution to piezoelectric inclusion, Dunn and Taya[6] estimated the effective properties using the dilute, self-consistent, Mori-Tanaka and differential micromechanical models. The present paper is concerned with deriving exact general solution for an infinite, anisotropic piezoelectric medium with an elliptic inclusion. The coupled electroelastic fields of the inclusion and matrix are given when the external elastic field and electric field are both constant. The developed theory is based on the central idea of the Stroh formalism established by Stroh[7] and further elaborated by Ting[8,9]. More recently, Stroh formalism was generalized to treat dislocations and line charges in linear piezoelectric media by Pak[10] and to solve the boundary value problems of lectroelastic media by Suo et al[11].

II. BASIC EQUATIONS

If no free charge and body force exist in piezoelectric body, the static elastic and electric field equations can be written as[12]

$$\partial_i D_i = 0 \tag{1}$$

Received 30 July 1993
* The project supported by the National Natural Science Foundation of China

$$\partial_i \sigma_{ij} = 0 \tag{2}$$

$$D_i = -\varepsilon_{is}\Phi, s + e_{irs}u_{r,s} \tag{3}$$

$$\sigma_{ij} = C_{ijrs}u_{r,s} + e_{eji}\Phi, s \tag{4}$$

and

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad E_i = -\Phi_{,i} \tag{5}$$

where $\boldsymbol{\gamma}, \boldsymbol{\sigma}, \boldsymbol{D}$ and $\boldsymbol{E}$ are strain, stress, electric displacement and electric field; $\boldsymbol{u}$ and $\boldsymbol{\Phi}$ the elastic displacement and electric potential, respectively. The elastic, piezoelectric and dielectric constants of the medium are represented by the fourth, third and second-order tensors $\boldsymbol{C}$, $\boldsymbol{e}$, $\boldsymbol{\varepsilon}$ respectively. Substitution of Eqs.(3) and (4) into Eqs.(1) and (2) yields

$$(C_{ijrs}u_r + e_{sji}\Phi)_{,si} = 0 \qquad (-\varepsilon_{is}\Phi + e_{irs}u_r)_{,si} = 0 \tag{6}$$

For two-dimensional problems in which u and Φ depend on x_1 and x_2 only, the general solution can be obtained by considering an arbitrary function of a linear combination of x_1 and x_2

$$\{u_r, \Phi\} = \boldsymbol{a} f(\xi_1 x_1 + \xi_2 x_2) \tag{7}$$

It is convenient, here and in the sequel, to take $\{u_r, \Phi\}$ to be a column with the entities indicated, so that $\boldsymbol{a}$ is likewise a four-component column. Without loss of generality, one can always take $\xi_1 = 1$, $\xi_2 = p$. Thus, the number p and the column $\boldsymbol{a}$ are determined by substituting Eq.(6) into (7), which gives

$$(C_{\alpha jr\beta}a_r + e_{\alpha j\beta}a_4)\xi_\alpha\xi_\beta = 0 \qquad (-\varepsilon_{\alpha\beta}a_4 + e_{\alpha r\beta}a_r)\xi_\alpha\xi_\beta = 0 \tag{8}$$

where $\alpha, \beta = 1$ or 2. This is an eigenvalue problem consisting of four equations; a nontrivial $\boldsymbol{a}$ exists if p is a root of the determinant polynomial. Since Eq.(8) admits no real root[11], the $\boldsymbol{p}_\alpha$ occur as four pairs of complex conjugates. We let

$$p_{\alpha+4} = \bar{p}_\alpha \qquad \mathrm{Im}(p_\alpha) > 0 \qquad \alpha = 1, 2, 3, 4 \tag{9}$$

where an overbar denotes the complex conjugate and Im stands for the imaginary part. More generally, we have

$$\boldsymbol{V} = \{u_r, \Phi\} = 2\mathrm{Re}\sum_{\alpha=1}^{4} \boldsymbol{a}_\alpha f_\alpha(z_\alpha) \tag{10}$$

in which Re stands for the real part, $\boldsymbol{a}_\alpha$ the associated columns, and $z_\alpha = x_1 + p_\alpha x_2$. For a given boundary value problem, the stress and the electric displacement obtained from Eqs.(4) and (10) are given by

$$\{\sigma_{2j}, D_2\} = 2\mathrm{Re}\sum_{\alpha=1}^{4} \boldsymbol{b}_\alpha f'_\alpha(z_\alpha) \qquad \{\sigma_{1j}, D_1\} = -2\mathrm{Re}\sum_{\alpha=1}^{4} \boldsymbol{b}_\alpha f'_\alpha(z_\alpha) \tag{11}$$

where, for a pair $(p, \boldsymbol{a})$, the associated $\boldsymbol{b}$ is

$$b_j = (C_{2jr\beta}a_r + e_{\beta j2}a_4)\xi_\beta \qquad b_4 = (-\varepsilon_{2\beta}a_4 + e_{2r\beta}a_r)\xi_\beta \tag{12}$$

Substituting Eq.(12) into (8) gives an alternative expression

$$b_j = -p^{-1}(C_{1jr\beta}a_r + e_{\beta j1}a_4)\xi_\beta \qquad b_4 = -p^{-1}(-\varepsilon_{1\beta}a_4 + e_{1r\beta}a_r)\xi_\beta \tag{13}$$

In matrix notation, the Eqs.(12) and (13) are expressed as

$$\boldsymbol{b} = (R^{\mathrm{T}} + pT)\boldsymbol{a} = -\frac{1}{p}(Q + pR)\boldsymbol{a} \tag{14}$$

where the superscript T stands for the transposed matrix and R, T, Q are 4×4 matrices given by

$$R = \begin{bmatrix} C_{1jr2} & e_{2j1} \\ e_{1r2}^{\mathrm{T}} & -\varepsilon_{12} \end{bmatrix}_{4\times4} \qquad Q = \begin{bmatrix} C_{1jr1} & e_{1j1} \\ e_{1r1}^{\mathrm{T}} & -\varepsilon_{11} \end{bmatrix}_{4\times4}$$

$$T = \begin{bmatrix} C_{2jr2} & e_{2j2} \\ e_{2r2}^{\mathrm{T}} & -\varepsilon_{22} \end{bmatrix}_{4\times4} \tag{15}$$

We see that Q and T are symmetric and T is positive definite, and the Eq.(14) can be recast in the standard eigenrelation

$$N\boldsymbol{\xi} = p\boldsymbol{\xi} \tag{16}$$

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^{\mathrm{T}} \end{bmatrix} \qquad \boldsymbol{\xi} = \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} \tag{17}$$

$$\left.\begin{aligned} &N_1 = -T^{-1}R^{\mathrm{T}} \qquad N_2 = T^{-1} = N_2^{\mathrm{T}} \\ &N_3 = RT^{-1}R^{\mathrm{T}} - Q = N_3^{\mathrm{T}} \end{aligned}\right\} \tag{18}$$

where N_2 and N_3 are also symmetric and N_2 is positive definite. If we define the 4×4 matrices A and B by

$$A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4) \qquad B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4) \tag{19}$$

each of the $\boldsymbol{a}$s is determined by the eigenvalue problem up to a complex-valued normalization constant. It can be proved that the matrices H, L, S introduced by Barnett and Lothe[13]

$$S = \mathrm{i}(2AB^{\mathrm{T}} - I) \qquad H = 2\mathrm{i}AA^{\mathrm{T}} \qquad L = -2\mathrm{i}BB^{\mathrm{T}} \tag{20}$$

where $\mathrm{i} = \sqrt{-1}$ and I is the unit matrix, are real and valid for the coupled electroelastic problem. By introducing the following auxiliary function

$$\boldsymbol{U} = 2\,\mathrm{Re}\sum_{\alpha=1}^{4} \boldsymbol{b}_\alpha f_\alpha(z_\alpha) \tag{21}$$

Eqs.(11) can be rewritten as

$$\{\sigma_{2j}, D_2\} = U_{,1} \qquad \{\sigma_{1j}, D_1\} = -U_{,2} \tag{22}$$

Finally, from Eqs.(10), (16) and (21), we have the following differential equation

$$\{V_{,2}\ , U_{,2}\} = N\{V_{,1}\ , U_{,1}\} \tag{23}$$

III. ELECTRIC AND ELASTIC FIELDS IN MATRIX AND INCLUSION

In an infinite anisotropic piezoelectric material, consider an elliptic inclusion whose boundary is given by

$$x_1 = a\cos\psi \qquad x_2 = b\sin\psi \tag{24}$$

where $2a, 2b$ are the major and minor axes of the ellipse. The inclusion is extended indefinitely in the x_3-direction, and the uniform stress and electric field are applied at infinity. The inclusion and the matrix have a perfect bonding along the interface (24). Let σ_{ij}^{∞}, γ_{ij}^{∞} be the stresses and strains, D_i^{∞}, E_s^{∞} the electric displacement and electric field in the matrix at infinity. They are defined from Eqs.(3) and (4)

$$\sigma_{ij}^{\infty} = C_{ijrs}\gamma_{rs}^{\infty} - e_{sji}E_s^{\infty} \qquad D_i^{\infty} = \varepsilon_{is}E_s^{\infty} + e_{irs}\gamma_{rs}^{\infty} \tag{25}$$

We note that σ_{ij}^{∞} have to be prescribed in such a way that $\gamma_{33}^{\infty} = 0$, the auxiliary functions $\boldsymbol{V}^{\infty}$ and $\boldsymbol{U}^{\infty}$ can be expressed using the variables in an infinite body

$$\boldsymbol{V}^{\infty} = \{u_r^{\infty}, \Phi^{\infty}\} = \{(x_1\gamma_1^{\infty} + x_2\gamma_2^{\infty}), (x_1E_1^{\infty} + x_2E_2^{\infty})\} \tag{26}$$

$$\boldsymbol{U}^{\infty} = \{(x_1\boldsymbol{t}_2^{\infty} - x_2\boldsymbol{t}_1^{\infty}), (x_1D_2^{\infty} - x_2D_1^{\infty})\} \tag{27}$$

in which

$$\left.\begin{aligned} \gamma_1^{\infty} &= \{\gamma_{11}^{\infty}, 0, 2\gamma_{13}^{\infty}\} = \boldsymbol{u}_{,1}^{\infty} \\ \gamma_2^{\infty} &= \{2\gamma_{21}^{\infty}, \gamma_{22}^{\infty}, 2\gamma_{23}^{\infty}\} = \boldsymbol{u}_{,2}^{\infty} \\ \boldsymbol{t}_1^{\infty} &= \{\sigma_{11}^{\infty}, \sigma_{12}^{\infty}, \sigma_{13}^{\infty}\} \\ \boldsymbol{t}_2^{\infty} &= \{\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}\} \end{aligned}\right\} \tag{28}$$

In engineering applications including the present case, f_1, f_2, f_3, f_4 have the same function form

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha) \qquad \alpha \text{ not summed} \tag{29}$$

where $q_\alpha, \alpha = 1,2,3,4$ are arbitrary complex constants. If we introduce the diagonal matrices

$$\boldsymbol{z} = \mathrm{diag}\{z_1, z_2, z_3, z_4\} \tag{30}$$

$$\boldsymbol{F}(z) = \mathrm{diag}\{f(z_1), f(z_2), f(z_3), f(z_4)\} \tag{31}$$

Eqs.(10) and (21) can be written as

$$\boldsymbol{V} = 2\mathrm{Re}\{AF(z)\boldsymbol{q}\} \tag{32}$$

$$\boldsymbol{U} = 2\mathrm{Re}\{BF(z)\boldsymbol{q}\} \tag{33}$$

in which $\boldsymbol{q}$ is the 4×1 matrix whose elements are q_α, $\alpha = 1,2,3,4$. Before we superimpose the general solution (32) and (33) onto (26) and (27), we replace the complex constant $\boldsymbol{q}$ by

$$\boldsymbol{q} = A^{\mathrm{T}}\boldsymbol{g} + B^{\mathrm{T}}\boldsymbol{h} \tag{34}$$

where $\boldsymbol{g}$ and $\boldsymbol{h}$ are real. We therefore consider the general solution

$$\boldsymbol{V} = \{u_r, \Phi\} = x_1\{\gamma_1^{\infty}, E_1^{\infty}\} + x_2\{\gamma_2^{\infty}, E_2^{\infty}\} + 2\mathrm{Re}\{AF(z)A^{\mathrm{T}}\}\boldsymbol{g} + 2\mathrm{Re}\{AF(z)B^{\mathrm{T}}\}\boldsymbol{h} \tag{35}$$

$$\boldsymbol{U} = x_1\{\boldsymbol{t}_2^\infty, D_2^\infty\} - x_2\{\boldsymbol{t}_1^\infty, D_1^\infty\} + 2\mathrm{Re}\{BF(z)A^{\mathrm{T}}\}\boldsymbol{g} + 2\mathrm{Re}\{BF(z)B^{\mathrm{T}}\}\boldsymbol{h} \tag{36}$$

of the auxiliary function for the piezoelectric materials with elliptic inclusion. Following Lekhnitskii[14], we choose the arbitrary function $f(z_\alpha)$ in $F(z)$ of the following form

$$\left.\begin{aligned} &F(z) = \mathrm{diag}(\xi_1^{-1}, \xi_2^{-1}, \xi_3^{-1}, \xi_4^{-1}) \\ &\xi_\alpha = \{z_\alpha + [z_\alpha^2 - (a^2 + p_\alpha^2 b^2)]^{1/2}\}/(a - \mathrm{i}p_\alpha b) \end{aligned}\right\} \tag{37}$$

It is clear that

$$\xi_\alpha^{-1} = \{z_\alpha - [z_\alpha^2 - (a^2 + p_\alpha^2 b^2)]^{1/2}\}/(a + \mathrm{i}p_\alpha b) \tag{38}$$

Along the interface (24), we then obtain

$$\xi_\alpha^{-1} = \cos\psi - \mathrm{i}\sin\psi \qquad F(z) = (\cos\psi - \mathrm{i}\sin\psi)I \tag{39}$$

We next consider the general solution of auxiliary functions $\boldsymbol{V}^0$ and $\boldsymbol{U}^0$ in the inclusion of piezoelectric ceramics. According to Wang[2], the coupled elastic and electric fields inside the inclusion stay uniform when the external elastic field and electric field are constant for the piezoelectric medium containing an elliptic inclusion

$$\boldsymbol{V}^0 = \{u_r^0, \Phi^0\} = \{(x_1\gamma_1^0 + x_2\gamma_2^0), (x_1E_1^o + x_2E_2^0)\} \tag{40}$$

$$\boldsymbol{U}^0 = \{(x_1\boldsymbol{t}_2^0 - x_2\boldsymbol{t}_1^0), (x_1D_2^0 - x_2D_1^0)\} \tag{41}$$

where

$$\left.\begin{aligned} &\gamma_1^0 = \{\gamma_{11}^0, \omega, 2\gamma_{13}^0\} = \boldsymbol{u}_{,1}^0 \\ &\gamma_2^0 = \{2\gamma_{21}^0 - \omega, \gamma_{22}^0, 2\gamma_{23}^0\} = \boldsymbol{u}_{,2}^0 \\ &\boldsymbol{t}_1^0 = \{\sigma_{11}^0, \sigma_{12}^0, \sigma_{13}^0\} \\ &\boldsymbol{t}_2^0 = \{\sigma_{21}^0, \sigma_{22}^0, \sigma_{23}^0\} \end{aligned}\right\} \tag{42}$$

in which ω is the rotation (counter clockwise) of the elliptic inclusion. The elastic and electric fields in the inclusion are also related by Eqs.(3) and (4)

$$\sigma_{ij}^0 = C_{ijrs}^0\gamma_{rs}^0 - e_{sji}^0E_s^0 \qquad D_i^0 = \varepsilon_{is}^0E_s^0 + e_{irs}^0\gamma_{rs}^0 \tag{43}$$

where C_{ijrs}^0, ε_{is}^0, e_{irs}^0 are the elastic constants, dielectric permittivity and piezoelectric constants of the inclusion, respectively. From the basic solution given by (35), (36) for the matrix and by (40), (41) for the inclusion, the problem reduces to the determination of the unknown constants $\boldsymbol{g}, \boldsymbol{h}$, $(\boldsymbol{t}_1^0, D_1^0)$, $(\boldsymbol{t}_2^0, D_2^0)$, (γ_1^0, E_1^0) and (γ_2^0, E_2^0) only.

Along the elliptic interface defined by (24), Eqs.(40) and (41) reduce to

$$\boldsymbol{V}^0 = a\cos\psi\{\gamma_1^0, E_1^0\} + b\sin\psi\{\gamma_2^0, E_2^0\} \tag{44}$$

$$\boldsymbol{U}^0 = a\cos\psi\{\boldsymbol{t}_2^0, D_2^0\} - b\sin\psi\{\boldsymbol{t}_1^0, D_1^0\} \tag{45}$$

while using (20) and (39), (35) and (36) become

$$\boldsymbol{V} = \cos\psi(a\{\gamma_1^\infty, E_1^\infty\} + \boldsymbol{h}) + \sin\psi(b\{\gamma_2^\infty, E_2^\infty\} - \boldsymbol{S}\boldsymbol{h} - \boldsymbol{H}\boldsymbol{g}) \tag{46}$$

$$\boldsymbol{U} = \cos\psi(a\{\boldsymbol{t}_2^\infty, D_2^\infty\} + \boldsymbol{g}) - \sin\psi(b\{\boldsymbol{t}_1^\infty, D_1^\infty\} - \boldsymbol{L}\boldsymbol{h} + S^{\mathrm{T}}\boldsymbol{g}) \tag{47}$$

Assume the bond is perfect, so that the displacement and potential, the stress and electric displacement are continuous across the bonded segment

$$\boldsymbol{V} = \boldsymbol{V}^0 \qquad \boldsymbol{U} = \boldsymbol{U}^0 \tag{48}$$

From Eqs.(23), (28), (42) and the solution to anisotropic elasticity problems introduced by Hwu and Ting[15], we have

$$\begin{bmatrix} D_1 & D_2 \\ -D_3 & D_1^{\mathrm{T}} \end{bmatrix} \{\boldsymbol{h}, \boldsymbol{g}\} = b\{\boldsymbol{d}_1, \boldsymbol{d}_2\} \tag{49}$$

where

$$D_1 = S + \frac{b}{a}N_1^0 \qquad D_2 = H + \frac{b}{a}N_2^0 \qquad D_3 = L - \frac{b}{a}N_3^0$$
$$\boldsymbol{d}_1 = (N_1 - N_1^0)\{\gamma_1^\infty, E_1^\infty\} + (N_2 - N_2^0)\{\boldsymbol{t}_2^\infty, D_2^\infty\}$$
$$\boldsymbol{d}_2 = (N_3 - N_3^0)\{\gamma_1^\infty, E_1^\infty\} + (N_1 - N_1^0)\{\boldsymbol{t}_2^\infty, D_2^\infty\}$$

Eq.(49) can be solved for $\boldsymbol{h}$ and $\boldsymbol{g}$ explicitly by inverse transformation, and we then obtain

$$\left.\begin{aligned} \boldsymbol{h} &= b(D_3 + D_1^{\mathrm{T}} D_2^{-1} D_1)^{-1}(D_1^{\mathrm{T}} D_2^{-1}\boldsymbol{d}_1 - \boldsymbol{d}_2) \\ \boldsymbol{g} &= b(D_2 + D_1 D_3^{-1} D_1^{\mathrm{T}})^{-1}(\boldsymbol{d}_1 + D_1 D_3^{-1}\boldsymbol{d}_2) \end{aligned}\right\} \tag{50}$$

We also rigorously prove that $(D_3+D_1^{\mathrm{T}}D_2^{-1}D_1)^{-1}$ and $(D_2+D_1D_3^{-1}D_1^{\mathrm{T}})^{-1}$ are both positive definite which justifies the existance of the inverse in Eqs.(50).

IV. IDENTICAL EQUATIONS

Let $\boldsymbol{n}(\omega), \boldsymbol{m}(\omega)$ be, respectively, the unit vectors tangent and normal to the interface, we have

$$\boldsymbol{n}^{\mathrm{T}}(\omega) = \{\cos\omega, \sin\omega, 0) \qquad \boldsymbol{m}^{\mathrm{T}}(\omega) = (-\sin\omega, \cos\omega, 0) \tag{51}$$

Therefore, Eqs.(15) can be generalized by

$$\left.\begin{aligned} R(\omega) &= \begin{bmatrix} C_{ijrs} & e_{sji} \\ e_{irs}^{\mathrm{T}} & -\varepsilon_{is} \end{bmatrix} n_i m_s \\ Q(\omega) &= \begin{bmatrix} C_{ijrs} & e_{sji} \\ e_{irs}^{\mathrm{T}} & -\varepsilon_{is} \end{bmatrix} n_i n_s \\ T(\omega) &= \begin{bmatrix} C_{ijrs} & e_{sji} \\ e_{irs}^{\mathrm{T}} & -\varepsilon_{is} \end{bmatrix} m_i m_s \end{aligned}\right\} \tag{52}$$

and it can be seen that (52) reduces to (15) when $\omega = 0$. Next we consider the generalized eigenrelation

$$N(\omega)\boldsymbol{\xi} = p(\omega)\boldsymbol{\xi} \tag{53}$$

$$N(\omega) = \begin{bmatrix} N_1(\omega) & N_2(\omega) \\ N_3(\omega) & N_1^{\mathrm{T}}(\omega) \end{bmatrix} \qquad \boldsymbol{\xi} = \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} \tag{54}$$

$$\left.\begin{aligned} &N_1(\omega) = -T^{-1}(\omega)R^{\mathrm{T}}(\omega) \qquad N_2(\omega) = T^{-1}(\omega) \\ &N_3(\omega) = R(\omega)T^{-1}(\omega)R^{\mathrm{T}}(\omega) - Q(\omega) \end{aligned}\right\} \tag{55}$$

It can be proved that the eigenvalues $p(\omega)$ are related to p in Eq.(18) by

$$p(\omega)=(p\cos\omega-\sin\omega)/(p\sin\omega+\cos\omega) \tag{56}$$

As before, Eqs.(53) have eight eigenvalues $p_\alpha(\omega), \mathrm{Im}(p_\alpha(\omega))>0,\ \alpha=1,2,3,4$ which come in four pairs of complex conjugates and can be combined into one compact form as

$$N(\omega)\left\{\begin{array}{c}A\\B\end{array}\right\}=\left[\begin{array}{cc}A & P(\omega)\\ B & P(\omega)\end{array}\right] \tag{57}$$

where A and B are defined in (19) and

$$\boldsymbol{P}(\omega)=\mathrm{diag}(p_1(\omega),\ p_2(\omega),\ p_3(\omega),\ p_4(\omega)) \tag{58}$$

Substituting $N(\omega)$ from (54), we obtain the identities

$$\left.\begin{array}{ll}2A\boldsymbol{P}(\omega)B^{\mathrm{T}} & =N_1(\omega)-\mathrm{i}[N_1(\omega)S-N_2(\omega)L]\\ & =N_1(\omega)-\mathrm{i}[SN_1(\omega)+HN_3(\omega)L]\\ 2A\boldsymbol{P}(\omega)A^{\mathrm{T}} & =N_2(\omega)-\mathrm{i}[N_1(\omega)H+N_2(\omega)S^{\mathrm{T}}]\\ 2B\boldsymbol{P}(\omega)B^{\mathrm{T}} & =N_3(\omega)-\mathrm{i}[N_3(\omega)S-N_1^{\mathrm{T}}(\omega)L]\end{array}\right\} \tag{59}$$

The proof parallels that of Barnett and Lothe[13] for the anisotropic elasticity problem, and we have an alternative expression for S,H,L defined in (20)

$$\left.\begin{array}{l}S=\dfrac{1}{\pi}\displaystyle\int_0^\pi N_1(\omega)\mathrm{d}\omega\\ H=\dfrac{1}{\pi}\displaystyle\int_0^\pi N_2(\omega)\mathrm{d}\omega\\ L=-\dfrac{1}{\pi}\displaystyle\int_0^\pi N_3(\omega)\mathrm{d}\omega\end{array}\right\} \tag{60}$$

to anisotropic piezoelectric materials. The three matrices S,H,L can be used to obtain the real-form solutions of the electroelastic fields in anisotropic piezoelectric medium, without determining the eigenvectors A,B.

V. FIELDS ALONG THE INTERFACE

Let $\boldsymbol{n}(\omega),\boldsymbol{m}(\omega)$ be, respectively, the unit vectors tangent and normal to the interface boundary, and T_m the stress and electric displacement vectors along the interface. We have

$$T_m=U_{,n}=\cos\omega U_{,1}+\sin\omega U_{,2} \tag{61}$$

Since $\boldsymbol{U}=\boldsymbol{U}^0$ at the interface, using (41) leads to

$$T_m=\cos\omega T_2^0-\sin\omega T_1^0 \tag{62}$$

where T^0 is the stress and the electric displacement vectors in the inclusion. Next consider the stress and the electric displacement vectors normal to the interface. Then

$$T_n=-U_{,m}=\sin\omega U_{,1}-\cos\omega U_{,2} \tag{63}$$

Substituting (36) into (63) leads to

$$T_n = \sin\omega T_2^\infty + \cos\omega T_1^\infty - 2Re\{BF_{,m}(z)A^{\mathrm{T}}\}\boldsymbol{g} - 2Re\{BF_{,m}(z)B^{\mathrm{T}}\}\boldsymbol{h} \tag{64}$$

in which T^∞ is the stress and the electric displacement vectors in the matrix at infinity. The differentiation of (38) and evaluation of the result at the interface (24) yields

$$\begin{aligned}\frac{\partial}{\partial m}\xi_\alpha^{-1} &= (p_\alpha\cos\omega - \sin\omega)\frac{\mathrm{d}}{\mathrm{d}z_\alpha}\xi_\alpha^{-1}\\ &= \left(\frac{1}{a}\cos\omega - \frac{\mathrm{i}}{\mathrm{b}}\sin\omega\right)p_\alpha(\omega)\end{aligned} \tag{65}$$

where we have made use of (56). Therefore, using (37) we obtain

$$F_{,m}(z) = \left(\frac{1}{a}\cos\omega - \frac{\mathrm{i}}{\mathrm{b}}\sin\omega\right)P(\omega) \tag{66}$$

where $P(\omega)$ is defined in (58). Finally, substituting (66) into (64) and using (59) yields

$$\begin{aligned}T_n(\omega) = {}& \cos\omega\{T_1^\infty - \frac{1}{a}[N_3(\omega)\boldsymbol{h} + N_1^{\mathrm{T}}(\omega)\boldsymbol{g}]\}\\ &+ \sin\omega\{T_2^\infty + \frac{1}{b}[N_3(\omega)(S\boldsymbol{h} + H\boldsymbol{g}) + N_1^{\mathrm{T}}(\omega)(S^{\mathrm{T}}\boldsymbol{g} - L\boldsymbol{h})]\}\end{aligned} \tag{67}$$

where Eqs.(67) are the real-form solution of the elastic and electric fields in an infinite piezoelectric medium with an embedded elliptic inclusion subject to a uniform stress at infinity. It is clear that the coupled fields are only dependent on the identities given by the elastic and electric constants and the boundary conditions. Finally, in the case of nonpiezoactive medium ($e_{sij} = 0$) where no coupled solution exists, particular formulas of independent elastic and electric fields can be derived from the general expressions (67), which are exactly the same as that given by Hwu and Ting[15] using the Stroh method in anisotropic elastic mechanics.

VI. CONCLUSION

In this paper, the Stroh method in anisotropic elastic mechanics was used to analyze the coupled elastic and electric fields in infinite piezoelectric medium containing an elliptical inclusion. The explicit real-form solutions for the electroelastic fields both inside the inclusion and on the boundary of the inclusion and matrix are obtained. The general expression can also be used in measuring the piezoelectrical constants of piezoelectric composites. It is apparent that understanding of the coupled electrical and mechanical properties of the generalized anisotropic piezoelectric body is essential to the design and manufacturing of piezoelectric components.

REFERENCES

[1] Sosa HA and Pak YE. Three-dimensional eigenfunction analysis of a crack in a piezoelectric material. *Int J Solids Structures*, 1990, 26: 1–15

[2] Wang B. Three-dimensional analysis of an ellipsodial inclusion in a piezoelectric material. *Int J Solids Structures*, 1992, 29: 293–308

[3] Wang B, Du Shanyi. Three-dimensional analysis of defects in a piezoelectric material. *Acta Mechanica Sinica*, 1992, 8(2): 181–185

[4] Chen T. The rotation of a rigid ellipsoidal inclusion embedded in an anisotropic piezoelectric medium. *Int J Solids Structures*, 1993, 30: 1983–1995

[5] Deeg WF. The analysis of dislocation, crack and inclusion problems in piezoelectric solids. Ph D Thesis, Stanford University, CA, 1980

[6] Dunn M and Taya M. Micromechanics predications of the effective electroelastic moduli of piezoelectric composites. *Int J Solids Structures*, 1993, 30: 161–175

[7] Stroh AN. Steady state problems in anisotropic elasticity. *J Math Phys*, 1962, 41: 77

[8] Ting TCT. Explicit solution and invariance of the singularities at an interface crack in anisotropic components. *Int J Solids Structures*, 1986, 22: 965–983

[9] Ting TCT. Some identities and the structure of Ni in the Stroh formalism of anisotropic elasticity.. *Q Appl Math*, 1988, 46: 109–120

[10] Pak YE. Linear electro-elastic fracture mechanics of piezoelectric materials. *Int J of Fracture*, 1992, 54: 79–100

[11] Suo Z, Kuo CM, Barnett DM and Willis JR. Fracture mechanics for piezoelectric ceramics. *J Mech Phys Solids*, 1992, 40: 739–765

[12] Maugin GA. Continum Mechanics of Electromagnetic Solids, North-Holland, Amsterdam, 1988

[13] Barnett DM and Lothe J. Synthesis of the sextic and the integral fomalism for dislocation, Green's function and surface waves in anisotropic elastic solids. *Phys Norv*, 1973, 7: 13–19

[14] Lekhnitskii GA. Anisotropic Plates. Gordon & Breach, NY, 1968

[15] Hwu Chyanbin and Ting TCT. Two-dimensional problems of the anisotropic elastic solid with an elliptic inclusion. *Q J Mech Appl Math*, 1989, 42: 553–572