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THE STRENGTH DISTRIBUTION OF BRITTLE MATERIALS WITH A HIGH CONCENTRATION OF CAVITIES

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Abstract—The maximum stress concentration factor in brittle materials with a high concentration of cavities is obtained. The interaction between the nearest cavities, in addition to the far field interactions, is taken into account to evaluate the strength distribution based on the statistical analysis of the nearest distance distribution. Through this investigation, it is found that the interaction between the nearest neighbors is much more important than the far field interactions, and one has to consider it in calculating the strength of brittle materials even if the volume fraction of cavities it contains is small. The other important conclusion is that the maximum stress concentration factor has a wide scattered distribution.

1. INTRODUCTION

It is a well known fact that the strength of brittle materials has a scattered distribution and this is due to the existence of randomly distributed defects in such materials. Since the distinguished work of Weibull, many researchers have been trying to establish the quantitative relation between the strength distribution of brittle materials and their random microstructures. They mainly considered the effects of size distribution and orientation distribution of defects [1, 2] and neglected their interactions.

It is extremely hard to obtain the stress field distribution in such materials by considering the defect interactions. Using the simple method developed by Mori and Tanaka [3], Tandon and Weng [4] derived the stress distribution in a material with a random distribution of inclusions. However, their result proved to be an approximation only by considering the far field interactions among inclusions. Therefore it may be a good method in calculating the effective elastic moduli, but inadequate to evaluate the local stress field fluctuation, which is the most important quantity in evaluating the strength distribution.

At the suggestion of McCoy and Beran's work [5], we not only consider the far field interactions but also take the interaction between the nearest inclusions into account to derive the distribution of the maximum stress in brittle materials with a high concentration of spherical cavities. First of all, the numerical distribution of cavities in volume V is assumed to be the binomial distribution, then the distribution function of the K nearest cavities to a reference cavity is derived. Based on the solution of an infinite matrix containing two cavities, the mean value and the standard deviation of the maximum stress concentration factor are obtained versus the volume fraction of cavities. Finally, the probability density function of the material strength is derived. Throughout this investigation, it is found that the interaction between the nearest neighbors is much more important than the far field interactions, and one has to consider this interaction in calculating the strength of brittle material even if the volume fraction of cavities it contains is small. The other important conclusion obtained is that for brittle material the wide scattered distribution of strength is mainly due to the wide scattered distribution of the maximum stress in such materials.

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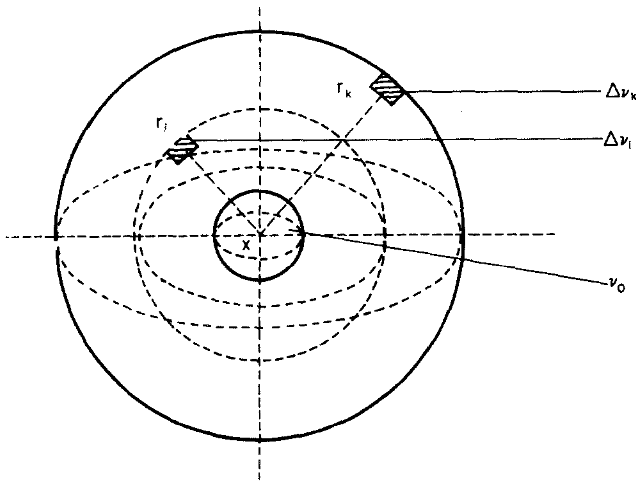

Fig. 1. Schematic of a random distribution of the nearest k inclusions.

2. A STATISTICAL MODEL OF THE RANDOM DISTRIBUTION OF SPHERICAL INCLUSIONS

The number of inclusions in volume V is assumed to be a random variable with a binomial distribution of probability. Thus the probability that the volume V contains i inclusions is given by

$$P(i, V) = C_n^i(1 - S)^{n-i}S^i, \quad i \leqslant n \tag{1}$$

where n is the maximum number of inclusions which may be embedded in V, and C_n^i is the binomial coefficient. The quantity S is given by

$$S = \int_V \lambda \, \mathrm{d}v/n, \tag{2}$$

where λ is the density function of the mean number of inclusions, which was introduced by Wang *et al.* [6]. Since the volume fraction corresponding to the maximum packing is 0.74, the maximum packing number of inclusions in volume V is given by

$$n = 0.74\, V/(\tfrac{4}{3}\pi a^3), \tag{3}$$

where a is the radius of each spherical inclusion. If the inclusions are distributed uniformly in volume V, i.e. $\lambda(=N)$ is a constant, which means the mean number in unit volume, eq. (1) gives the same result as the one given by Herczynski [7]. The Poisson distribution function is also contained in eq. (1) when the spheres become points [8]. Through eq. (1), we can obtain the conditional probability density function locating the nearest k neighbors to a reference inclusion positioned at the origin. First of all, the volume V containing the nearest k inclusions is divided into v_0; v_1, Δv_1; v_2, Δv_2; $\ldots$; v_k, Δv_k (Fig. 1) to satisfy the requirement that $\mathbf{r}_1 \in \Delta v_1, \mathbf{r}_2 \in \Delta v_2, \ldots, \mathbf{r}_k \in \Delta v_k$, and there is no inclusion within v_i ($i = 0, 1, \ldots, k$), in which $\mathbf{r}_1$, $\mathbf{r}_2, \ldots, \mathbf{r}_k$ are positions of the k nearest neighbors, and $v_0, v_1, \ldots, v_k$ are determined by

$$\begin{aligned}
v_0 &= \tfrac{4}{3}\pi(2a)^3 \\
v_1 &= \tfrac{4}{3}n(r_1^3 - 8a^3) - \Delta v_1 \\
v_2 &= \tfrac{4}{3}\pi(r_2^3 - r_1^3) - \Delta v_2 \\
&\vdots \\
v_k &= \tfrac{4}{3}\pi(r_k^3 - r_{k-1}^3) - \Delta v_k.
\end{aligned} \tag{4}$$

Table 1. Experimental results of the metallographic observations

		Group 1	Group 2	Group 3	Group 4
	C_f	0.4776	0.4882	0.4730	0.5646
	$E(a)$	0.0446	0.02684	0.0340	0.02855
	$D(a)$	0.0050	0.0044	0.0030	0.0040
	N	1289	6031	2872	5791
$D(N)$	Experimental	349	2115	634	1883
	Theoretical	457	2052	1036	1372

$E(a)$ and $D(a)$ are the mean value and variance of the particle radius, and N is the mean number of particles in unit volume; $D(N)$ is the variance of the number of particles in unit volume. The last row in Table 1 gives the theoretical results predicted by eq. (1).

Since the event $\{\mathbf{r}_1 \in \Delta v_2, \ldots, \mathbf{r}_k \in \Delta v_k; r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k\}$ is equivalent to the event $\{i = 0, v_1; i \geqslant 1, \Delta v_1; i = 0, v_2; i \geqslant 1, \Delta v_2; \ldots; i \geqslant 1, \Delta v_k\}$ we obtain the probability

$$\begin{aligned} &P\{\mathbf{r}_1 \in \Delta v_1, \mathbf{r}_2 \in \Delta v_2, \ldots, \mathbf{r}_k \in \Delta v_k; r_1 \leqslant r_2 \leqslant \ldots \leqslant r_k\} \\ &\quad = P\{i = 0, v_1; i \geqslant 1, \Delta v_1; i = 0, v_2; i \geqslant 1, \Delta v_2; \ldots; i \geqslant 1, \Delta v_k\} \\ &\quad = P(i = 0, v_1)P(i \geqslant 1, \Delta v_1) \ldots P(i = 0, v_k)P(i \geqslant 1, \Delta v_k). \end{aligned} \tag{5}$$

Substitution of eq. (1) into eq. (5) yields

$$\begin{aligned} &P\{r_1 \in \Delta v_1, \mathbf{r}_2 \in \Delta v_2, \ldots, \mathbf{r}_k \in \Delta v_k; r_1 \leqslant r_2 \leqslant r_3 \leqslant \ldots \leqslant r_k\} \\ &\quad = \prod_{i=1}^{k} \left(1 - \frac{\int_{v_i} \lambda \, \mathrm{d}v}{m v_i}\right)^{m v_i} \left[1 - \left(1 - \frac{\int_{\Delta v_i} \lambda \, \mathrm{d}v}{m \, \Delta v_i}\right)^{m \Delta v_i}\right], \end{aligned} \tag{6}$$

where m is the maximum packing number in unit volume, and is given by

$$m = 0.74/(\tfrac{4}{3}\pi a^3). \tag{7}$$

Under the condition that there is an inclusion at the origin, the probability in eq. (6) becomes

$$\begin{aligned} &P\{\mathbf{r}_1 \in \Delta v_1, \mathbf{r}_2 \in \Delta v_2, \ldots, \mathbf{r}_k \in \Delta v_k; r_1 \leqslant r_2 \leqslant \ldots \leqslant r_k)|(i = 0, v_0)\} \\ &\quad = \left(1 - \frac{\int_{v_0} \lambda \, \mathrm{d}v}{m v_0}\right)^{-m v_0} \prod_{i=1}^{k} \left(1 - \frac{\int_{v_i} \lambda \, \mathrm{d}v}{m v_i}\right)^{m v_i} \left[1 - \left(1 - \frac{\int_{\Delta v_i} \lambda \, \mathrm{d}v}{m \, \Delta v_i}\right)^{m \Delta v_i}\right]. \end{aligned} \tag{8}$$

The conditional probability density function of the k nearest positions can be derived by using eq. (8), and is given by

$$\begin{aligned} f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k) &= \lim_{\max|\Delta v_i| \to 0} \frac{P\{\mathbf{r}_1 \in \Delta v_1, \mathbf{r}_2 \in \Delta v_2, \ldots, \mathbf{r}_k \in \Delta v_k; r_1 \leqslant r_2 \leqslant \ldots \leqslant r_k)|(i = 0, v_0)\}}{\Delta v_1 \, \Delta v_2 \ldots \Delta v_k} \\ &= -m^k \left(1 - \frac{\int_{v_0} \lambda \, \mathrm{d}v}{m v_0}\right)^{m v_0} \prod_{i=1}^{k} \log\left(1 - \frac{\lambda_i}{m}\right)\left(1 - \frac{\int_{v_i} \lambda \, \mathrm{d}v}{m v_i}\right)^{m v_i}. \end{aligned} \tag{9}$$

If the inclusions are uniformly distributed in the matrix, eq. (9) becomes

$$f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k) = -m^k \log^k\left(1 - \frac{N}{m}\right)^{\frac{4}{3}\pi m(r_k^3 - 8a^3)}. \tag{10}$$

Therefore, the probability density function of the nearest neighbor is determined by

$$f(\mathbf{r}_1) = -m \log\left(1 - \frac{N}{m}\right)\left(1 - \frac{N}{m}\right)^{\frac{4}{3}\pi m(r_1^3 - 8a^3)}, \tag{11}$$

from which the probability density function of the nearest distance between two inclusions can be obtained through

$$f(\mathbf{r}_1) = \int_0^{2\pi} \mathrm{d}\Phi \int_0^{\pi} f(\mathbf{r}_1) r_1^2 \sin\theta \, \mathrm{d}\theta = -4\pi m r_1^2 \log\left(1 - \frac{N}{m}\right)\left(1 - \frac{N}{m}\right)^{\frac{4}{3}\pi m(r_1^3 - 8a^3)}, \tag{12}$$

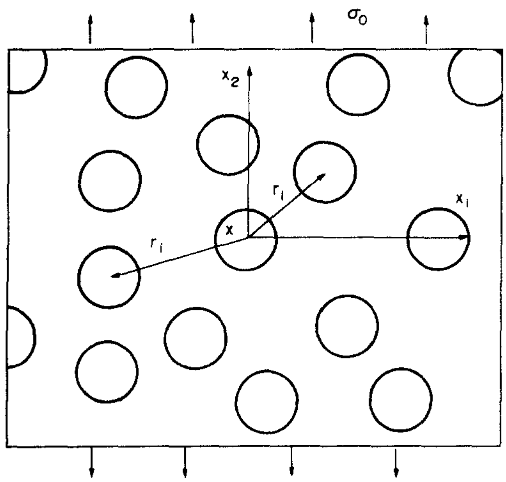

Fig. 2. Schematic of the material element.

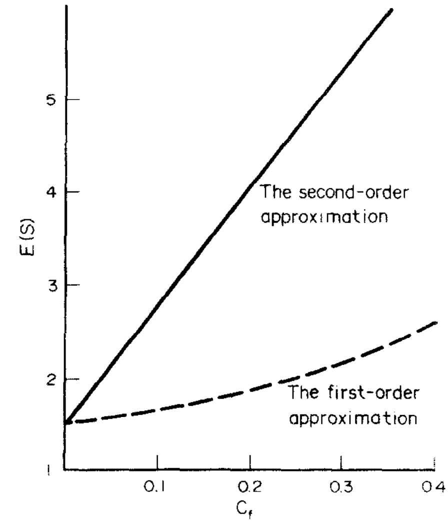

Fig. 3. The mean value of the maximum stress concentration factor.

which will be used to derive the distribution function of the maximum stress concentration factor by considering the interaction between the nearest inclusions.

To check the assumption of eq. (1), quantitative metallographic observation experiments have been done on cast iron. Our attention focuses on the numerical distribution and size distribution of graphite particles. The results are shown in Table 1.

3. THE STATISTICAL PROPERTY OF THE MAXIMUM STRESS CONCENTRATION FACTOR AND THE STRENGTH DISTRIBUTION OF BRITTLE MATERIALS WITH A HIGH CONCENTRATION OF CAVITIES

Brittle materials with a high concentration of cavities are studied in this paper, an element of which is shown in Fig. 2.

The main aim of this research is to obtain the maximum stress concentration factor, which is a random variable. Since it is impossible to derive the solution by considering all the interactions among cavities, the following approximate procedure will be adopted. (a) A typical region which contains k cavities is taken into account, and only these k cavities are assumed to exist in the infinite matrix. The effect of all surrounding cavities is reflected by the average stress $\langle\sigma_m\rangle$ in the matrix, by which the k cavities are loaded instead of σ_0. (b) The solution of an infinite matrix containing k cavities is derived. (c) The statistical property of the maximum stress concentration factor is obtained based on the above discussion. When the region Ω contains only one cavity, the maximum stress concentration factor can be easily obtained and denoted as the first-order approximation, when two cavities are taken into account, the second-order approximation is obtained, etc.

3.1. *The first-order approximation*

Wang *et al.* [2] have obtained the average stress in the matrix containing a random distribution of cavities, which is given by

$$\langle\sigma_{ij}^M\rangle = \frac{1}{1-C_f}\sigma_{ij}^0, \tag{13}$$

where σ_{ij}^0 is the external stress field. Under equal triaxial tension, the maximum stress is given by

$$\sigma_{\max} = \frac{A}{3(1-C_f)}\sigma_{kk}^0. \tag{14}$$

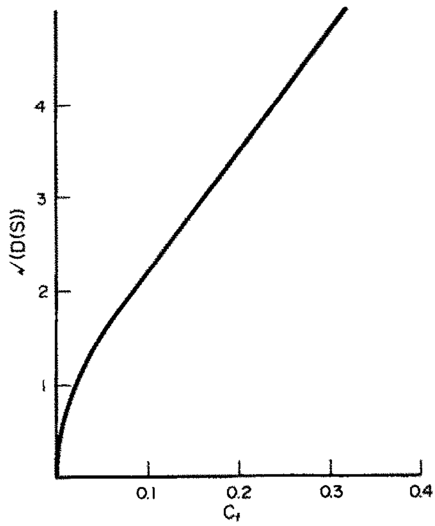

Fig. 4. The standard deviation of the maximum stress concentration factor.

If Poisson's ratio equals 0.25, A equals 1.5. Therefore, for the first-order approximation, the maximum stress concentration factor is

$$S = \frac{1.5}{1 - C_f}, \tag{15}$$

which is shown in Fig. 3 versus the volume fraction C_f of cavities.

3.2. *The second-order approximation*

To obtain the second-order approximation, one has to derive the solution of an infinity containing two cavities. Based on the finite element solution obtained by Rodin and Hwang (private communication), the maximum stress can be expressed as

$$\begin{aligned} \sigma_{\max} &= p(r/a)\langle\sigma^M\rangle \\ &= \frac{p(r/a)}{1 - C_f}\cdot\tfrac{1}{3}\sigma^0_{kk}, \end{aligned} \tag{16}$$

where $p(r/a)$ is the infinite stress concentration factor containing two cavities, and r is the distance between two cavity centers. Since r and a are random variables, the maximum stress $\sigma_{\max}$ is also a random variable. The mean value and standard deviation are derived as follows.

If the probability density function of each cavity radius is assumed to be $f(a)$, by combining eq. (12), the joint probability density function of r and a is given by

$$f(r, a) = f(r\,|\,a)f(a) = -4\pi m r^2 f(a)\log\left(1 - \frac{N}{m}\right)\left(1 - \frac{N}{m}\right)^{\frac{4}{3}\pi m(r^3 - 8a^3)}. \tag{17}$$

Thus, the average value of the maximum stress concentration factor is determined by

$$\begin{aligned} E(S) &= \int_0^\infty \mathrm{d}a \int_{2a}^\infty \frac{p\left(\frac{r}{a}\right)}{1 - C_f} f(r, a)\,\mathrm{d}r = -\frac{2.22}{1 - C_f}\log\left(1 - \frac{C_f}{0.74}\right) \\ &\quad \times \left(1 - \frac{C_f}{0.74}\right)^{-5.92}\int_2^\infty \left(1 - \frac{C_f}{0.74}\right)^{0.74Y^3} Y^2 p(Y)\,\mathrm{d}Y, \end{aligned} \tag{18}$$

in which the relations $4/3\,\pi a^3 m = 0.74$, $N = C_f/(4/3\pi a^3)$ and $Y = r/a$ are used. The second-order moment is given by

$$E(S^2) = -\frac{2.22}{(1 - C_f)^2}\log\left(1 - \frac{C_f}{0.74}\right)\left(1 - \frac{C_f}{0.74}\right)^{-5.92}\int_2^\infty \left(1 - \frac{C_f}{0.74}\right)^{0.74Y^3} Y^2 p^2(Y)\,\mathrm{d}Y. \tag{19}$$

By combining eqs (18) and (19), the standard deviation is given by

$$\sqrt{[D(S)]} = \sqrt{[E(S^2) - E^2(S)]}. \tag{20}$$

When the Poisson's ratio equals 0.25, the mean value and the standard deviation of the maximum stress concentration factor are as shown in Figs 3 and 4, respectively.

According to the calculated results, we can conclude that: (a) to calculate the maximum stress concentration, or to analyse the strength distribution of brittle material with cavities, one has to consider the interaction between the nearest cavities, even for the case with a small concentration of cavities; (b) the standard deviation of the maximum stress concentration factor increases rapidly with increasing volume fraction of cavities, which may be the main cause why the strength of brittle materials has a widely scattered distribution.

4. THE STRENGTH DISTRIBUTION FUNCTION

Based on the distribution function of the maximum stress concentration factor obtained above, the strength distribution function can be derived by using the weakest-link hypothesis. If the failure probability P of a single cavity is denoted by

$$P = P_r(S\sigma_0 \geqslant \sigma_{cr}), \tag{21}$$

where S is the maximum stress concentration factor and σ_{cr} is the critical failure stress, the failure probability of volume V, which contains NV cavities, is given by

$$P_f = 1 - (1 - P)^{NV}. \tag{22}$$

Usually, NV is quite large and P is rather small, whereas NVP is always finite. In such cases, eq. (22) becomes

$$P_f = 1 - \lim_{P \to 0} [(1 - P)^{1/P}]^{NVP}$$
$$= 1 - \exp(-NVP). \tag{23}$$

If the cavities are distributed non-uniformly in the matrix, the failure probability is given by

$$P_f = 1 - \exp\left(-\int_V \lambda P \,\mathrm{d}v\right), \tag{24}$$

where λ is the numerical density function introduced above.

5. CONCLUDING REMARKS

In this paper, the statistical properties of the maximum stress concentration factor are obtained for brittle materials with a high concentration of cavities. Through the above analysis, the main results can be concluded as follows.

(1) The binomial distribution function is used to describe the numerical distribution of cavities, through which the position distribution function of the k nearest cavities to a reference cavity is obtained. Since the number density function λ is introduced, this model can be used to describe non-uniform distributions of inclusions.

(2) The mean value of the maximum stress concentration factor is obtained by considering not only the far field interactions of cavities but also the interaction between the nearest two voids, and it is found that the local field fluctuation contributes greatly to the stress concentration.

(3) The standard deviation of the maximum stress concentration factor is quite large, and increases rapidly with increasing volume fraction of cavities. This may be one of the most important factors for explaining the wide scattered property of strength. If the interactions among cavities are neglected and we further assume that defects in brittle materials are spherical voids and the size of each void is so small that the whole specimen can be treated as infinite, the strength does not scatter at all. However, according to this investigation, the maximum stress concentration factor has a very large scattered distribution; therefore the strength must also have a large scattered distribution.

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