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THE CRACK-BRIDGING MODEL WITH THE CONSIDERATION OF RESIDUAL STRESS IN PARTICULATE-REINFORCED CERAMICS

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Abstract—The crack-bridging model with the consideration of residual stress in particulate-reinforced ceramics is presented. The compressive residual stress in the matrix is obtained through microstructural analysis. In this paper, the bridging particles are treated as randomly distributed discrete springs, and the bridging stress distribution with the consideration of residual stress is derived. A rough estimation on toughening in Al_2O_3/Al composites gives reasonable results.

INTRODUCTION

THE FRACTURE toughness of ceramics can be greatly enhanced by dispersed ductile metal particles. This phenomenon can be comprehended by the mechanism of crack-bridging. The crack-bridging model presumes that the faces of an advancing crack in the ceramic are pinned together by intact particles for some distance behind the crack tip, reducing the crack-tip stress intensity that would otherwise occur. Rose[1] and Budiansky *et al.*[2] treated the bridging particles as a continuous distribution of spring and investigated the effect of these particles on the toughness. Along such a way, it is difficult to consider the effect of residual stress which is created by the misfit of thermal expansion coefficient between metal inclusions and the matrix. Sigl *et al.*[3] presented some observations in WC/Co and Al_2O_3/Al composites, and the presence of crack bridging was confirmed. But they found that the level of toughening predicted was appreciably less than the measured toughening. The disparity can be partially due to the effect of residual stress. The present investigation attempts to incorporate the residual stress in the crack-bridging model. The bridging particles are treated as randomly distributed discrete springs. It is found that the compressive residual stress in the matrix has a great effect on toughening.

CALCULATION OF RESIDUAL STRESS

The composite considered consists of a ceramic matrix containing a uniform distribution of inclusions. Since in many of the composite systems of interest, the inclusions have a large thermal expansion coefficient than the matrix, residual tension thus exist in the inclusions and residual compression in the ceramic when the composite is cooled. The average residual stress within an inclusion or in the matrix can be derived as follows:

$$C_f\langle\sigma^I\rangle + (1 - C_f)\langle\sigma^M\rangle = 0 \tag{1}$$

where, C_f is the volume fraction of inclusions, and $\langle\sigma^I\rangle, \langle\sigma^M\rangle$ are the average residual stresses in the inclusion and matrix, respectively. Equation (1) means that the residual stress is self-equilibrium. In order to establish the relation between $\langle\sigma^I\rangle$ and $\langle\sigma^M\rangle$, it is assumed that, every single inclusion with a misfitting strain ϵ^T is in the shape of a sphere, and is loaded by $\langle\sigma^M\rangle$ (Fig. 1).

This assumption is very similar to the self-consistent approach. To obtain the average stress within an inclusion, one can use $\langle\sigma^M\rangle$ to reflect the influence of the other inclusions, just like in self-consistent approach, the effective moduli are used. The next step is to solve a single inclusion problem under the coupled actions of external stress field $\langle\sigma^M\rangle$ and the misfitting strain ϵ^T. This solution has derived by Wang[4] as follows:

$$(\mathbf{I} + \mathbf{D} : \mathbf{B}^I) : \langle\sigma^I\rangle = \langle\sigma^M\rangle + \mathbf{D} : \epsilon^T \tag{2}$$

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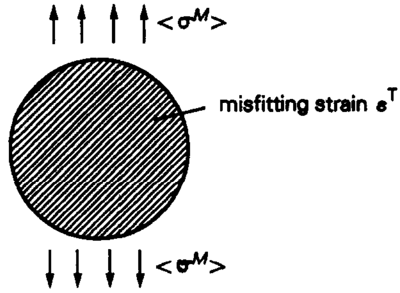

Fig. 1. A spherical inclusion under $[\sigma^M]$ and $[\epsilon^T]$.

where **I** is the identity tensor, and

$$D_{ijkl} = D_1 E^1_{ijkl} + D_2 E^2_{ijkl} \tag{3}$$

$$B^1_{klmn} = B^1_1 E^1_{klmn} + B^1_2 E^2_{klmn} \tag{4}$$

where

$$D_1 = \frac{2G_m}{15}\frac{7-5\gamma_m}{1-\gamma_m}, \qquad D_2 = \frac{2G_m}{15}\frac{5\gamma_m+1}{1-\gamma_m} \tag{5}$$

$$B^1_1 = \frac{1}{2G} - \frac{1}{2G_m}, \qquad B^1_2 = \frac{\gamma_m}{E_m} - \frac{\gamma}{E} \tag{6}$$

$$E^1_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \qquad E^2_{ijkl} = \delta_{ij}\delta_{kl}, \tag{7}$$

E_m, G_m, γ_m are Young's modulus, shear modulus and Poisson ratio of the matrix, and E, G, γ are the corresponding elastic constants of the inclusion.

Substitution of eq. (1) into eq. (2) gives

$$\langle \sigma^I \rangle = (1 - C_f)[\mathbf{I} + (1 - C_f)\mathbf{D} : \mathbf{B}^1]^{-1} : \mathbf{D} : \epsilon^T \tag{8}$$

$$\langle \sigma^M \rangle = -C_f[\mathbf{I} + (1 - C_f)\mathbf{D} : \mathbf{B}^1] : \mathbf{D} : \epsilon^T. \tag{9}$$

According to the contraction rule of the basic tensors E^1, E^2, i.e.

$$\mathbf{E}^1 : \mathbf{E}^1 = \mathbf{E}^1, \quad \mathbf{E}^1 : \mathbf{E}^2 = \mathbf{E}^2 : \mathbf{E}^1 = \mathbf{E}^2, \quad \mathbf{E}^2 : \mathbf{E}^2 = 3\mathbf{E}^2, \tag{10}$$

one obtains

$$\langle \sigma^I \rangle = (1 - C_f)[H_1 D_1 \mathbf{E}^1 + (H_1 D_2 + H_2 D_1 + 3H_2 D_2)\mathbf{E}^2] : \epsilon^T \tag{11}$$

$$\langle \sigma^M \rangle = -C_f[H_1 D_1 \mathbf{E}^1 + (H_1 D_2 + H_2 D_1 + 3H_2 D_2)\mathbf{E}^2] : \epsilon^T \tag{12}$$

where

$$H_1 = \frac{1}{1 + (1 - C_f)\beta_1(7 - 5\gamma_m)} \tag{13}$$

$$H_2 = -\frac{(1 - C_f)[\beta_1(5\gamma_m + 1) + 10\beta_2(1 + \gamma_m)]}{[1 + \beta_1(1 - C_f)(7 - 5\gamma_m)]^2 + 3(1 - C_f)[1 + \beta_1(1 - C_f)(7 - 5\gamma_m)][\beta_1(5\gamma_m + 1) + 10\beta_2(1 + \gamma_m)]}$$

$$\beta_1 = \frac{1}{15}\left(\frac{G_m}{G} - 1\right)\Big/(1 - \gamma_m), \quad \beta_2 = \frac{1}{15}\left(\gamma_m - \frac{\gamma E_m}{E}\right)\Big/(1 - \gamma_m). \tag{14}$$

For composites whose constituents have different thermal expansion coefficients, the misfitting strain can be expressed as

$$\epsilon^T_{ii} = (\alpha - \alpha_m)\,\Delta T = \Delta\alpha\,\Delta T, \tag{15}$$

where $ii = 11, 22, 33$ (no summation), and the other components are zero, α, α_m are thermal expansion coefficient of inclusions and matrix, respectively. ΔT is the cooling range. Substitution of eq. (15) into eqs (11) and (12) gives

$$\langle \sigma_{ii}^I \rangle = (1 - C_f)\,\Delta\alpha\,\Delta T[H_1 D_1 + 3(H_1 D_2 + H_2 D_1 + 3H_2 D_2)] \tag{16}$$

$$\langle \sigma_{ii}^M \rangle = -C_f \Delta\alpha\,\Delta T[H_1 D_1 + 3(H_1 D_2 + H_2 D_1 + 3H_2 D_2)] \tag{17}$$

where $ii = 11, 22, 33$ (no summation).

For Al_2O_3/Al composite system, the material properties are

$$E_m = 20\,\text{GPa}, \quad \gamma_m = 0.2, \quad E = 70\,\text{GPa}, \quad \gamma = 0.33$$

$$\Delta\alpha = 1.1 \times 10^{-5}, \quad \Delta T = 500°\text{C}, \quad C_f = 0.2. \tag{18}$$

Substitution of the material properties (18) into eqs (16) and (17) gives

$$\langle \sigma_{ii}^I \rangle = 754.58\,\text{MPa}, \quad \langle \sigma_{ii}^M \rangle = -188.64\,\text{MPa}. \tag{19}$$

CALCULATION OF BRIDGING STRESS

The randomly distributed inclusions can be considered as springs which restrict the crack opening (Fig. 2). The bridging stress means the stress in the spring. For the case of linear springs we adopt the notation of Rose[1] here, and write the spring stress, or bridging stress as

$$\sigma(x) = \frac{kE_m u}{1 - \gamma_m^2} \tag{20}$$

in terms of the crack-face displacement u, Young's modulus E_m, Poisson's ratio γ_m and a spring stiffness coefficient k.

According to Appendix B, the reduction of the crack-face displacement due to the action of the residual stress in the matrix is

$$u_2 = \frac{4(1-\gamma_m^2)}{\pi E_m}\langle \sigma_{11}^M \rangle \int_0^L \log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}}\right) \mathrm{d}x' - \sum_{i=1}^{N_L} \frac{4(1-\gamma_m^2)}{\pi E_m} \mathrm{d}\langle \sigma_{11}^M \rangle \log\left(\frac{\sqrt{x}+\sqrt{x_i}}{\sqrt{|x-x_i|}}\right) \tag{21}$$

where N_L is the number of inclusions in $(0, L)$, d is the average diameter of the inclusion and x_i is the center of the i-th inclusion. Since N_L and x_i are random variables and u_2 is also a random variable. The reduction of the crack-face displacement due to the action of the bridging stress is

$$\begin{aligned} u_3 &= \sum_{i=1}^{N_L} \frac{4(1-\gamma_m^2)}{\pi E_m} \int_{x_i-(d/2)}^{x_i+(d/2)} \sigma(x')\log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}}\right) \mathrm{d}x' \\ &\doteq \sum_{i=1}^{N_L} \frac{4(1-\gamma_m^2)}{\pi E_m} \sigma(x_i)\,\mathrm{d}\log\left(\frac{\sqrt{x}+\sqrt{x_i}}{\sqrt{|x-x_i|}}\right) \end{aligned} \tag{22}$$

where $\sigma(x_i)$ is the bridging stress of the i-th inclusion. In the same reason, u_3 is also a random variable. In order to derive the average values of u_2 and u_3, the following assumptions are introduced.

Assumptions:

(1) On the interval $[l_{i-1}, l_i]$, the number of inclusions obeys the Poisson distribution with parameter λ, i.e. for $n = 0, 1, 2, \ldots,$

$$P_r[N_{\Delta i} = n] = (n!)^{-1}\left(\int_{l_{i-1}}^{l_i} \lambda\,\mathrm{d}x\right)^n \exp\left(-\int_{l_{i-1}}^{l_i} \lambda\,\mathrm{d}x\right) \tag{23}$$

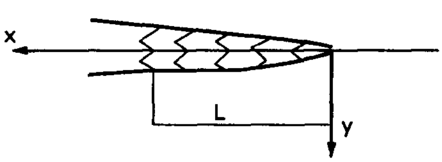

Fig. 2. A schematic illustrating the bridging process.

where $\Delta i = l_i - l_{i-1}$, $P_r[N_{\Delta i} = n]$ is the probability of $N_{\Delta i} = n$, and λ means the average number of inclusions in unit length.

(2) $\{N_{\Delta i}\}$ has independent increments on the intervals with a restriction for the intersection of them, i.e. for $\Delta i = \Delta 1, \Delta 2, \ldots, \Delta k$

$$P_r[N_{\Delta 1} = n_1, N_{\Delta 2} = n_2, \ldots, N_{\Delta k} = n_k] = \prod_{i=1}^{k} P_r[N_{\Delta i} = n_i]. \tag{24}$$

(3) $P_r[N_0 = 0] = 1$, which means that there is no inclusion at $x = 0$. The random terms in eqs (21) and (22) can be expressed as

$$u_p = \sum_{i=1}^{N_L} u(x, x_i) \tag{25}$$

where $u(x, x_i)$ is the function of x and x_i. The average u_p can be derived with the aid of above assumptions.

$$\begin{aligned}
\langle u_p \rangle &= \sum_{n=0}^{\infty} P_r[N_L = n] \left\langle \sum_{i=1}^{N_L} u(x, x_i) \,|N_L = n \right\rangle \\
&= \sum_{n=1}^{\infty} (n!)^{-1} \left(\int_0^L \lambda \, \mathrm{d}x \right)^n \exp\left(-\int_0^L \lambda \, \mathrm{d}x \right) n \left(\int_0^L \lambda \, \mathrm{d}x \right)^{-1} \int_0^L \lambda u(x, x_i) \, \mathrm{d}x_i \\
&= \int_0^L \lambda u(x, x_i) \, \mathrm{d}x_i \left[\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\int_0^L \lambda \, \mathrm{d}x \right)^{n-1} \exp\left(-\int_0^L \lambda \, \mathrm{d}x \right) \right] \\
&= \int_0^L \lambda u(x, x_i) \, \mathrm{d}x_i .
\end{aligned} \tag{26}$$

In derivation of eq. (26), the probability density function of the independent random variable x_i, $i = 1, 2 \ldots, N_L$ takes the form

$$p(x_i) = \left(\int_0^L \lambda \, \mathrm{d}x \right)^{-1} \lambda(x_i) \tag{27}$$

which is the consequence of the assumptions.

With the aid of eq. (26), the crack-face displacement equation can be expressed as

$$\begin{aligned}
\frac{(1-\gamma_m^2)\sigma(x)}{kE_m} &= \frac{4(1-\gamma_m^2)K/\sqrt{x}}{E_m\sqrt{2\pi}} - \frac{4(1-\gamma_m^2)\,\mathrm{d}}{\pi E_m} \int_0^L \lambda \sigma(x') \log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}} \right) \mathrm{d}x' - \frac{4(1-\gamma_m^2)}{\pi E_m} \\
&\quad \times \langle \sigma_{11}^M \rangle \int_0^L \log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}} \right) \mathrm{d}x' + \frac{4(1-\gamma_m^2)\,\mathrm{d}}{\pi E_m} \langle \sigma_{11}^M \rangle \int_0^L \lambda \log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}} \right) \mathrm{d}x'
\end{aligned} \tag{28}$$

where K is the stress intensity factor. If the inclusions are uniformly distributed, λ is a constant m, which means the average number of inclusions in unit length. One can approximately obtain

$$C_f = md \tag{29}$$

where C_f is the volume fraction of inclusions. Substitution of eq. (29) into eq. (28) yields

$$\begin{aligned}
\frac{(1-\gamma_m^2)\sigma(x)}{kE} &= \frac{4(1-\gamma_m^2)K\sqrt{x}}{E_m/\sqrt{2\pi}} - \frac{4(1-\gamma_m^2)}{\pi E_m} C_f \int_0^L \sigma(x') \log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}} \right) \mathrm{d}x' \\
&\quad - (1-C_f) \frac{4(1-\gamma_m^2)}{\pi E_m} \langle \sigma_{11}^M \rangle \int_0^L \log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}} \right) \mathrm{d}x',
\end{aligned} \tag{30}$$

or in non-dimensional form this is

$$g(s) + C_f \int_0^\alpha g(t) \log\left(\frac{\sqrt{s}+\sqrt{t}}{\sqrt{|s-t|}} \right) \mathrm{d}t + (1-C_f) G(\sigma_{11}^M) \int_0^\alpha \log\left(\frac{\sqrt{s}+\sqrt{t}}{\sqrt{|s-t|}} \right) \mathrm{d}t = \sqrt{s} \tag{31}$$

where

$$s = \frac{4kx}{\pi}, \quad t = \frac{4kx'}{\pi}, \quad g = \frac{\sigma}{K\sqrt{2k}}, \quad G(\sigma_{11}^M) = \frac{\langle \sigma_{11}^M \rangle}{K\sqrt{2k}}, \quad \alpha = \frac{4kL}{\pi}.$$

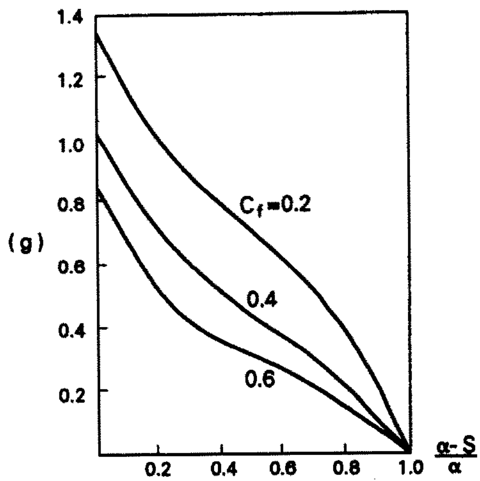

Fig. 3. Non-dimensional bridging stress distribution with $\alpha = 5, G(\sigma_{11}^M) = 0$.

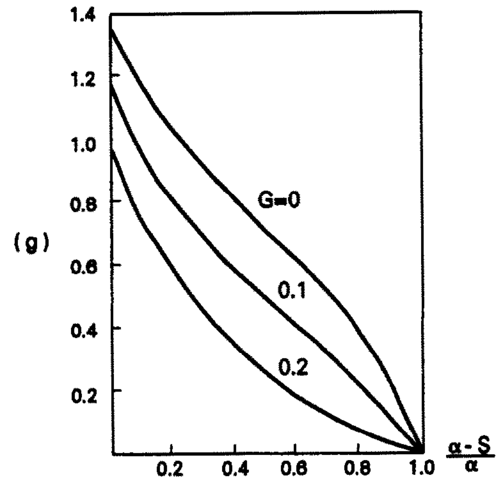

Fig. 4. Non-dimensional bridging stress distribution with $\alpha = 5, C_f = 0.2$.

If $C_f = 1$, which correspond to the continuous distribution of springs, eq. (31) becomes the same form as the one derived by Budiansky *et al.*[2]. A similar procedure to Budiansky *et al.* is used to solve the integral eq. (31), which is described in detail in Appendix A. Under the different volume fraction of inclusions and the different residual stress, the non-dimensional bridging stress distribution is obtained, respectively, and shown in Figs 3 and 4.

CALCULATION OF TOUGHENING

The compressive residual stress in the matrix $\langle \sigma_{11}^M \rangle$ and the bridging stress $\sigma(x)$ contribute to the toughening of ceramics. The inclusions can be treated as a random distribution of concentrated forces, which can reduce the stress intensity factor ΔK, that follows as

$$\Delta K_1 = -\sqrt{\frac{2}{\pi}} \sum_{i=1}^{N_L} \int_{x_i-(d/2)}^{x_i+(d/2)} \frac{\sigma(x)}{\sqrt{x}}\, \mathrm{d}x \doteq -\sqrt{\frac{2}{\pi}} \sum_{i=1}^{N_L} \frac{\sigma(x_i)}{\sqrt{x_i}}\, \mathrm{d} \tag{32}$$

where $\sigma(x_i)$ is the bridging stress of the i-th inclusion. The contribution of the compressive residual stress in the matrix is

$$\Delta K_2 = \sqrt{\frac{2}{\pi}} \int_0^L \frac{\langle \sigma_{11}^M \rangle}{\sqrt{x}}\, \mathrm{d}x - \sqrt{\frac{2}{\pi}} \sum_{i=1}^{N_L} \int_{x_i-(d/2)}^{x_i+(d/2)} \frac{\langle \sigma_{11}^M \rangle}{\sqrt{x}}\, \mathrm{d}x$$

$$\doteq \frac{2\sqrt{2L}}{\sqrt{\pi}} \langle \sigma_{11}^M \rangle - \sqrt{\frac{2}{\pi}}\, \mathrm{d} \langle \sigma_{11}^M \rangle \sum_{i=1}^{N_L} \frac{1}{\sqrt{x_i}}. \tag{33}$$

According to eq. (26), the average values of ΔK_1 and ΔK_2 can be derived as

$$\langle \Delta K_1 \rangle = -\sqrt{\frac{2}{\pi}}\, \mathrm{d} \int_0^L \lambda \frac{\sigma(x)}{\sqrt{x}}\, \mathrm{d}x \tag{34}$$

$$\langle \Delta K_2 \rangle = \frac{2\sqrt{2L}}{\sqrt{\pi}} \langle \sigma_{11}^M \rangle - \sqrt{\frac{2}{\pi}}\, \mathrm{d} \langle \sigma_{11}^M \rangle \int_0^L \frac{\lambda}{\sqrt{x}}\, \mathrm{d}x. \tag{35}$$

The total toughening effect can be obtained as

$$\Delta K = \langle \Delta K_1 \rangle + \langle \Delta K_2 \rangle$$

$$= -\sqrt{\frac{2}{\pi}}\, \mathrm{d} \int_0^L \lambda \frac{\sigma(x)}{\sqrt{x}}\, \mathrm{d}x + \frac{2\sqrt{2}}{\sqrt{\pi}} \langle \sigma_{11}^M \rangle \left(\sqrt{L} - \frac{d}{2} \int_0^L \frac{\lambda}{\sqrt{x}}\, \mathrm{d}x \right). \tag{36}$$

If the inclusions are uniformly distributed, i.e. λ is a constant m,

$$\Delta K = -\sqrt{\frac{2}{\pi}}\, C_f \int_0^L \frac{\sigma(x)}{\sqrt{x}}\, dx + \frac{2\sqrt{2L}}{\sqrt{\pi}} \langle \sigma_{11}^M \rangle (1 - C_f). \tag{37}$$

Since available experimental information is very limited, a rough estimation of the toughening effect of Al_2O_3/Al composite system is given as follows. The bridging stress $\sigma(x)$ is taken as a constant, and equals to the yield stress σ_Y of aluminum particles, which may correspond to rigid–plastic springs. The eq. (37) becomes

$$\Delta K = -2\frac{\sqrt{2L}}{\sqrt{\pi}} [C_f \sigma_Y - (1 - C_f)\langle \sigma_{11}^M \rangle]. \tag{38}$$

By using the experimental results given by Sigl *et al.*[3], i.e.

$$L = 80\,\mu\text{m}, \quad C_f = 0.2, \quad \sigma_Y = 70\,\text{MPa}, \tag{39}$$

substitution of eq. (19) into eq. (38) yields

$$\Delta K = -2.354\,\text{MPa}\sqrt{m}, \tag{40}$$

whereas the experimental result is

$$\Delta K_e = -5.4\,\text{MPa}\sqrt{m}. \tag{41}$$

CONCLUDING REMARKS

The present research attempts to develop a crack-bridging model with the consideration of residual stress. The metal particles in ceramics are treated as randomly distributed discrete springs. In case of a uniform distribution of particles, the average residual stress distribution is derived, and the effect of particulate volume fraction and residual stress on toughening is investigated.

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APPENDIX A: INTEGRAL-EQUATION SOLUTION

A similar method to Budiansky *et al.*[2] is used to derive the solution of eq. (31). A convenient set-up for solving the integral eq. (31) is obtained by differentiation, which gives

$$g's + \tfrac{1}{2}C_f \int_0^\alpha \sqrt{\frac{t}{s}} \frac{g(t)}{t - s}\, dt + \tfrac{1}{2}(1 - C_f)G \int_0^\alpha \sqrt{\frac{t}{s}} \frac{dt}{t - s} = \frac{1}{2\sqrt{s}}. \tag{A1}$$

If we now let

$$g(s) = \frac{H(s)}{\sqrt{s}} \tag{A2}$$

then A(1) reduces to

$$2sH(s) - H(s) + C_f s \int_0^\alpha \frac{H(t)}{t - s}\, dt = s - (1 - C_f)sG\left[2\sqrt{\alpha} + \sqrt{s}\log\left|\frac{\sqrt{\alpha} - \sqrt{s}}{\sqrt{a} + \sqrt{s}}\right|\right]. \tag{A3}$$

A neat way to proceed is to write

$$H(s) = A\left(\frac{s}{\alpha}\right) + f(s) \tag{A4}$$

where $A = \sqrt{\alpha}g(\alpha)$ and $f(0) = f(\alpha) = 0$. Then

$$A\left[(\tfrac{1}{2} + C_f)s + \frac{C_f s^2}{\alpha}\log\frac{\alpha - s}{s}\right] + 2sf'(s) - f(s) + C_f s\int_0^\alpha \frac{f(t)}{t-s}\,\mathrm{d}t = s - (1-C_f)sG\left[2\sqrt{\alpha} + \sqrt{s}\ln\left|\frac{\sqrt{\alpha}-\sqrt{s}}{\sqrt{\alpha}+\sqrt{s}}\right|\right], \tag{A5}$$

and with

$$s = \frac{\alpha}{2}(1 - \cos\theta), \tag{A6}$$

the Fourier-expansion

$$f = \sum_{n=1}^{N} \alpha_n \sin(n\theta) \tag{A7}$$

is appropriate. Since

$$\int_0^\alpha \frac{f(t)}{t-s}\,\mathrm{d}t = n\sum_{n=1}^{N} \alpha_n \cos(n\theta), \tag{A8}$$

eq. (A5) becomes

$$A\left[(\tfrac{1}{2} + C_f)\frac{\alpha}{2}(1-\cos\theta) + \tfrac{1}{4}C_f\alpha(1-\cos\theta)^2\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right] + 2\left(\frac{1-\cos\theta}{\sin\theta}\right)\sum_{n=1}^{N} n\alpha_n\cos n\theta$$

$$- \sum_{n=1}^{N}\alpha_n\sin n\theta + \frac{\pi}{2}C_f\alpha(1-\cos\theta)\sum_{n=1}^{N}\alpha_n\cos n\theta$$

$$= \frac{\alpha}{2}(1-\cos\theta) - \frac{\alpha}{2}(1-\cos\theta)(1-C_f)G\left[2\sqrt{\alpha} + \sqrt{a}\sin\frac{\theta}{2}\log\left(\frac{1-\sin\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}\right)\right] \tag{A9}$$

If we now multiply (A9) by $\sin(m\theta)$ and integrate over $(0, \pi)$, we get

$$C_m A - \frac{\pi}{2}\sum_{n=1}^{N} D_{mn}\alpha_n = B_m \tag{A10}$$

where

$$C_m = (1 + \alpha C_f)F_m + \frac{\alpha C_f}{4}T_m$$

$$F_m = \begin{cases} \dfrac{1}{m} & (m \text{ odd}) \\ -\dfrac{m}{m^2-1} & (m \text{ even}) \end{cases}$$

$$D_{mn} = \begin{cases} 2\alpha C_f P_{mn} & m < n \\ 1 + 2n + 2\alpha C_f P_{mn} & m = n \\ 4n(-1)^{m+n} + 2\alpha C_f P_{mn} & m > n \end{cases}$$

$$P_{mn} = \begin{cases} -\dfrac{m}{m^2-n^2} & (m+n \text{ odd}) \\ \dfrac{m(m^2-n^2-1)}{(m^2-n^2-1)^2-4n^2} & (m+n \text{ even}) \end{cases}$$

$$T_m = \int_0^\pi (1-\cos\theta)^2\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\sin(m\theta)\,\mathrm{d}\theta$$

$$B_m = [\alpha - 2\alpha\sqrt{\alpha}(1-C_f)G]F_m - \frac{\alpha^{3/2}}{2\sqrt{2}}(1-C_f)GX_m$$

$$X_m = \int_0^\pi (1-\cos\theta)^{3/2}\log\left(\frac{1+\sin(\theta/2)}{1+\sin(\theta/2)}\right)\sin(m\theta)\,\mathrm{d}\theta.$$

The substitution of (A4) into the undifferential integral eq. (31), and its assertion at the point $s = \alpha$ leads to

$$P_0 A + C_f\sqrt{\alpha}\sum_{n=1}^{N} P_n\alpha_n = Z \tag{A11}$$

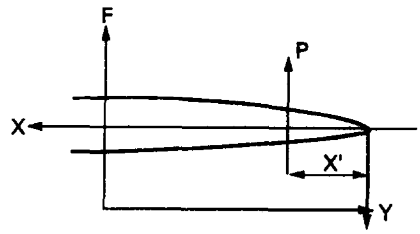

Fig. B1. Schematic of a semi-infinite plane crack.

where

$$P_0 = \frac{1}{\sqrt{\alpha}} + \tfrac{1}{3}C_f\sqrt{\alpha}(2\log 2 + 1)$$

$$Z = \sqrt{\alpha} - (1 - C_f)\alpha G$$

$$P_n = \int_0^\pi \cos\frac{\theta}{2}\log\left(\frac{1+\sin(\theta/2)}{1-\sin(\theta/2}\right)\sin(n\theta)\,d\theta.$$

Eqs (A10) and (A11) constitute $N+1$ linear equations for A and α_n $(n = 1, 2 \ldots, N)$, and can be solved numerically.

APPENDIX B: THE OPENING DISPLACEMENT CREATED BY A CONCENTRATED FORCE ON THE CRACK SURFACE

According to the displacement formula (Tade *et al.*[5]), the crack-face displacement created by a concentrated force on the crack surface can be expressed as

$$\delta = \frac{2}{E'}\int_0^\alpha K_{IP}\frac{\partial K_{IF}}{\partial F}\,d\zeta \tag{B1}$$

where

$$E' = \begin{cases} E & \text{for plane stress} \\ \dfrac{E}{1-\gamma^2} & \text{for plane stress} \end{cases} \tag{B2}$$

K_{IP} and K_{IF} are the stress intensity factors created by the concentrated force P and the imaginary force F, respectively. According to Tada *et al.*[5], for a semi-infinite plane crack, one knows

$$K_{IP} = \frac{p}{\sqrt{2\pi\zeta}}, \quad K_{IF} = \frac{F}{\sqrt{2\pi(\zeta+\Delta)}} \tag{B3}$$

where

$$\Delta x - x'. \tag{B4}$$

Substitution of eq. (B3) into eq. (B1) yields

$$\delta = \frac{2P(1-\gamma^2)}{\pi E}\log\left(\frac{\sqrt{x}+\sqrt{x'}}{|\sqrt{x}-\sqrt{x'}|}\right) = \frac{4P(1-\gamma^2)}{\pi E}\log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}}\right)$$

$$= \frac{4P(1-\gamma^2)}{\pi E}\log\left(\frac{\sqrt{x}+\sqrt{x'}}{\sqrt{|x-x'|}}\right).$$

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