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PROBABILITY FRACTURE MECHANICS OF BRITTLE MATERIALS WITH RANDOM-DISTRIBUTED DEFECTS UNDER MULTI-AXIAL STATES OF STRESS

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Abstract—Due to the presence of structural defects, the strength of brittle material has a substantial dispersion. This can be described by Weibull distribution function under uniaxial state of stress. The parameters in Weibull distribution function can be obtained through defect mechanics analysis, and the existed analyses are neglected in the microdefects interaction. In this paper, the distribution function is derived under multi-axial state of stress, and the parameters in it are obtained through microscopic analysis considering microdefects interaction.

INTRODUCTION

BRITTLE materials are characterized by a substantial dispersion in the fracture strength. This is due to the presence of structural defects and to the dominant role that these defects play in the failure process. This circumstance implies that the size, shape, orientation and the distribution of defects must be included to explain their rupture behavior.

A statistical distribution function of strength proposed by Weibull revealed itself as a useful tool in correlating experimental results under uniaxial states of stress. It takes the form

$$F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{1}$$

where σ_0 is a scale parameter, m is a shape parameter and σ is the fracture strength. Similarily, when brittle material is loaded by multi-axial stress, the distribution function of strength is[1],

$$F(\lambda, \psi) = 1 - \exp\left[-\left(\frac{\lambda}{\beta}\right)^\alpha\right] \tag{2}$$

where, the applied stress state is defined by a stress vector $\sigma_i = \lambda\psi_i$, λ is the stress intensity, ψ_i defines the stress mode.

Equations (1) and (2) are phenomenological models, and the parameters in them can be obtained through experiments. Quite a few authors try to derive these parameters by microstructure analysis. In this paper, the defects in brittle materials are idealized as random distributed microcracks. The stress field created by this group of microcracks was first calculated, then the distribution function of fracture strength for multi-axial states of stress was derived using maximum statistics. The difference of mean strength between the case of neglecting crack interaction and the case of considering crack interaction is calculated.

BASIC THEORY

1. *Calculation of stress field*

The brittle material is considered to contain random distributed microcracks and loaded by σ^0_{ij}. According to Kunin's discussion[2], the stress perturbation created by the α-th crack is

$$\Delta\sigma_{ij}(x) = \int_\Omega S_{ijkl}(\mathbf{x} - \mathbf{x}')E(n_k u_l)\, \mathrm{d}\Omega(\mathbf{x}') \tag{3}$$

where n_k is the unit normal vector of crack surface and u_l is the opening of the crack. Symbol $E(\ \ )$ denotes the mean value with respect to crack orientation. $S_{ijkl}(\mathbf{x} - \mathbf{x}')$ is the fourth order symmetric tensor, and it's components are shown in Appendix I. The stress perturbation created by N microcracks is,

$$\Delta\sigma_{ij}(\mathbf{\dot{x}}) = \sum_{\alpha=1}^{N} \Delta\sigma_{ij}(\mathbf{x})$$

$$= -\sum_{\alpha=1}^{N} \int_{\Omega} S_{ijkl}(\mathbf{x}-\mathbf{x}')E(n_k u_l)\,\mathrm{d}\Omega. \tag{4}$$

The mean stress perturbation in the matrix is[3]

$$\langle\Delta\sigma_{ij}(\mathbf{x})\rangle = -\int_{v-v_0} \rho(\mathbf{x}) \int_{\Omega} S_{ijkl}(\mathbf{x}-\mathbf{x}')E(n_k u_l)\,\Omega\,\mathrm{d}v(\mathbf{x}), \tag{5}$$

where V_0 is the region in which only one crack can exist and $\rho(\mathbf{x})$ is the crack density. If the cracks are uniformly distributed, $\rho(\mathbf{x}) = m$ (const). In such case, it yields,

$$\langle\Delta\sigma_{ij}(\mathbf{x})\rangle = -mE(n_k u_l) \int_{v-v_0} \mathrm{d}v(\mathbf{x}) \int_{\Omega} S_{ijkl}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\Omega. \tag{6}$$

If the crack is a penny shape with it's radius, V_0 a sphere. From[3], one obtains,

$$\int_{v-v_0} \mathrm{d}v(\mathbf{x}) \int_{\Omega} S_{ijkl}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\Omega = \pi a^2 D_{ijkl} \tag{7}$$

where

$$D_{ijkl} = D_1 E^1_{ijkl} + D_2 E^2_{ijkl}$$

$$D_1 = \left(\frac{2}{15}\right)\left(\frac{7-5\gamma_0}{1-\gamma_0}\right)G_0, \quad D_2 = \left(\frac{2}{15}\right)\left(\frac{5\gamma_0+3}{1-\gamma_0}\right)G_0 \tag{8}$$

where γ_0, G_0 are the Poisson ratio and shear modulus of the material. Values of E^1_{ijkl} and E^2_{ijkl} are shown in the Appendix.

Substitution of eq. (7) into eq. (6) gives, the mean stress in the matrix.

$$\sigma^m_{ij} = \sigma_{ij} - m\pi a^2 D_{ijkl} E(n_k u_l). \tag{9}$$

It is assumed that, every crack is loaded by σ^m_{ij} with a random distribution of orientation, $E(n_k u_l)$ can be expressed as follows[3]

$$E(n_k u_l) = H_{klpq}\langle\sigma^m_{pq}\rangle, \tag{10}$$

where

$$H_{klpq} = (H_l E^1_{klpq} + H_2 E^2_{klpq})\frac{\langle a^3\rangle}{\pi a^2}$$

$$H_1 = \frac{32(1-\gamma_0^2)(5-\gamma_0)}{45(2-\gamma_0)E_0} \tag{11}$$

$$H_2 = -\frac{16(1-\gamma_0^2)}{45(2-\gamma_0)E_0}.$$

Substitution eq. (10) into eq. (9) yields,

$$\sigma^m_{ik} = A_{ijkl}\sigma^0_{kl} = (A_1 E_{ijkl} + A_2 E^2_{ijkl})\sigma^0_{kl} \tag{12}$$

where

$$A_1 = \frac{1}{1-fD_1H_1}$$

$$A_2 = -\frac{f(D_2H_1 + D_1H_2 + 3D_2H_2)}{(1-fD_1H_1)^2 + 3(1-fD_1H_1)(D_2H_2 + D_1H_2 + 3D_2H_2)f} \tag{13}$$

where $f = m\langle a^3\rangle$.

2. *A statistical theory for the fracture of brittle materials*

A large number of experiments revealed that the fracture strength of brittle materials under uniaxial state of stress obeys the Weibull distribution. Assuming that the fracture of brittle materials depend on the group of microcracks discussed above, and the probability of failure of brittle material for a given state of stress coincides with the probability that it contains at least one crack at, or beyond, the critical state, as a consequence of extremum distribution theory[4], the strength distribution function for a single crack is,

$$g(\sigma_0) = C(\sigma_0 - \varepsilon)^k \tag{14}$$

where C, ε, k are constants of material.

According to eq. (12), the mean stress in a brittle solid subjected to uniaxial state of stress is,

$$\langle \sigma_{11}^m \rangle = \langle \sigma_{22}^m \rangle = A_{1133}\sigma_{33}^0, \tag{15}$$

$$\langle \sigma_{33}^m \rangle = A_{3333}\sigma_{33}^0.$$

The other components are zero. The stress state is completely defined by,

$$\sigma_i = \lambda\psi_i \tag{16}$$

where ψ_i is the stress mode, λ is the stress intensity, and,

$$\lambda = \sqrt{2A_{1133}^2 + A_{3333}^2}\,(\sigma_{33}^0) = A\sigma_{33}^0. \tag{17}$$

Substitution of eq. (17) into eq. (14) yields,

$$g(\sigma_0) = C\left(\frac{\lambda}{A} - \varepsilon\right)^k$$

$$= C'(\lambda - \varepsilon')^k \tag{18}$$

where $C' = CA^{-k}, \varepsilon' = A\varepsilon$.

From eq. (18), it is found that C', ε' not only depend on the material, but also depend on the stress mode. According to eq. (12). It is known that, if a brittle solid is subjected to triple state of stress, the mean stress in the matrix is also a triple state of stress. Therefore, we can consider that eq. (18) is suitable for any triple state of stress, and only with different parameters C', ε'. According to extremum statistics[4], the strength distribution function, as a consequence of eq. (18), is

$$F(\lambda, \psi) = 1 - \exp\left[-\left(\frac{\lambda}{\beta}\right)^{2\alpha}\right] \tag{19}$$

where, β depends on the stress mode ψ.

In conclusion, we have testified that the generalized Weibull distribution function equation (19) is correct for a multi-axial state of stress.

Assumption: the fracture criterion for a single crack is,

$$S \leqslant S^* \tag{20}$$

where S^* is the critical value of S for a single crack.

We can assume that $S = \lambda_I^2\Phi(\psi_I)$ without losing generality, λ_I is the internal stress intensity for a microcrack, ψ_I is the internal stress mode.

If the probability $P_R(\mathrm{d}\Omega)$ that the condition $S \geqslant S^*$ occurs for at least one crack having unit normal vector inside the solid angle $\mathrm{d}\Omega$, can be written,

$$P_R(\mathrm{d}\Omega) = P(S)\,\mathrm{d}\Omega. \tag{21}$$

The probability $P_S(\mathrm{d}\Omega)$ of the opposite occurrence ($S < S^*$ for every crack with unit normal vector in $\mathrm{d}\Omega$) will be,

$$P_S(\mathrm{d}\Omega) = 1 - P_R(S)\,\mathrm{d}\Omega = \exp[-P(S)\,\mathrm{d}\Omega]. \tag{22}$$

So, the probability of $S < S^*$ for all cracks (probability of survival) is obtained by the Weakest Link Theory,

$$P_S = \exp\left[-\int_{\Omega} P(S)\,\mathrm{d}\Omega\right]. \tag{23}$$

and, therefore, the probability of rupture is

$$P_R = 1 - \exp\left[-\int_{\Omega} P(S)\,\mathrm{d}\Omega\right]. \tag{24}$$

Equation (24) expresses a general form of the rupture probability function, which is derived from the microstructure theory. The form of $P(s)$, determining in its turn the functional form of P_R, can be derived from theoretical arguments, based on its physical interpretation, or from the experimental macroscopic evidence. The second way of reasoning is followed here and a distribution, taking the form as eq. (19), is assumed to fit, with reasonable accuracy, the experimental results for at least a particular stress state. The following work of this paper is to derive the value of parameter β through combining eq. (19) and eq. (24).

If eq. (19) is valid for a stress mode ψ_0, i.e.

$$F(\lambda_{00}, \psi_0) = 1 - \exp\left[-\left(\frac{\lambda_{00}}{\beta_0}\right)^{2\alpha_0}\right] \tag{25}$$

where, λ_{00} is the external stress intensity. By comparing eq. (25) with eq. (24) we obtain,

$$P(S_0)\,\mathrm{d}\Omega = \frac{\lambda_{00}^{2\alpha_0}}{\beta^{2\alpha_0}}, \tag{26}$$

where, $S_0 = \lambda_{0i}^2 \Phi_{0i}$, λ_{0i} is the internal stress intensity of a single crack, which is associated to the external field (λ_{00}, ψ_0). Let

$$B_0 = \frac{\lambda_{0i}^2}{\lambda_{00}^2} \Phi_{0i}$$

so,

$$S_0 = \lambda_{00}^2 B_0. \tag{27}$$

The solution of eq. (26) is obtained as,

$$P(S_0) = \frac{S_0^{\alpha_0}}{\beta_0^{2\alpha_0} \int_{\Omega} B_0^{\alpha_0}\,\mathrm{d}\Omega}. \tag{28}$$

So, if the internal stress field is

$$S = \lambda_1 \Phi_1 = \lambda_0^{2\alpha_0} B^{\alpha_0}, \quad B = \frac{\lambda_1^2}{\lambda_0^2} \Phi_1$$

for any state of stress, then,

$$P(S) = \frac{S^{\alpha_0}}{\beta_0^{2\alpha_0} \int_{\Omega} B_0^{\alpha_0}\,\mathrm{d}\Omega} \tag{29}$$

where, λ_0 is the external stress intensity for this state of stress. By substituting eq. (29) into eq. (24) and comparing with eq. (19), one obtains,

$$\beta(\psi) = \beta_0 \left[\frac{\int B_0^{\alpha_0}\,\mathrm{d}\Omega}{\int B^{\alpha_0}\,\mathrm{d}\Omega}\right]^{\frac{1}{2\alpha_0}} \tag{30}$$

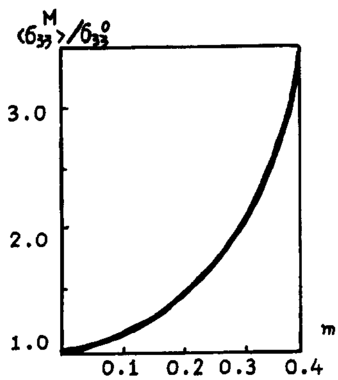

Fig. 1. The mean stress $\sigma_{33}^m/\sigma_{33}^0$ in the matrix.

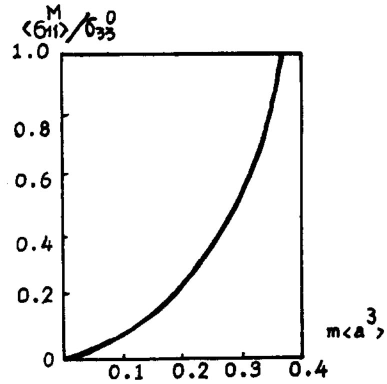

Fig. 2. The mean stress $\sigma_{11}^m/\sigma_{33}^0$ in the matrix.

By substituting

$$B = \frac{\lambda_i^2}{\lambda_0^2}\Phi_i,\ B_0 = \frac{\lambda_{0i}^2}{\lambda_{00}^2}\Phi_{0i}$$

in eq. (30), the form of parameter $\beta(\psi)$ is obtained.

So, the expectation of strength for multi-axial states of stress is,

$$E[\lambda] = \beta(\psi)\Gamma\left(1 + \frac{1}{2\alpha_0}\right). \tag{31}$$

The variance is,

$$D[\lambda] = \beta^2(\psi)\left\{\Gamma\left(1 + \frac{1}{\alpha_0}\right) - \left[\Gamma\left(1 + \frac{1}{2\alpha_0}\right)\right]^2\right\} \tag{32}$$

where Γ is the gamma function.

CALCULATING RESULTS AND DISCUSSION

(1) The calculation of stress field in a cracked solid. The interesting problem is to calculate the effective field, which a single crack, in a brittle solid with random crack, is subjected to. According to eq. (15), the results are plotted vs $m\langle a^3\rangle$ in Figs. 1 and 2.

(2) The difference of mean strength between the case of neglecting crack interaction and the case of considering crack interaction.

$$R(\times 100\%) = \left(\frac{E(\lambda) - E(\lambda^*)}{E(\lambda)}\right) \times 100\% \tag{33}$$

where, $E(\lambda)$, $E(\lambda^*)$ are the expectations of strength with neglecting and considering crack interactions, respectively. If the difference of the stress modes for two cases is neglected and

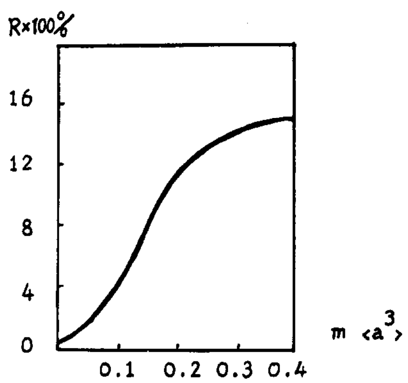

Fig. 3. The error of prediction for mean strength, neglecting crack interaction.

recalling that $\lambda_{0i} = \lambda_{00}$, $\lambda_0 = \lambda_i$ for the case of neglecting the crack interactions,

$$R = \left(1 - \frac{\lambda_{00} \cdot \lambda_i}{\lambda_{0i} \cdot \lambda_0}\right) \times 100\%. \tag{34}$$

In the case of equibiaxial tension ($\sigma_{11} = \sigma_{33} = 0$), the ratio R is plotted vs $m\langle a^3\rangle$ in Fig. 3.

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APPENDIX

$$S_{ijkl}(\mathbf{x} - \mathbf{x}') = (2\mu_0 - a_1)E^1_{ijkl} + (\lambda_0 - a_2)E^2_{ijkl} - a_3 E^3_{ijkl} - a_4 E^4_{ijkl} - a_5 E^5_{ijkl} - a_6 E^6_{ijkl}, \tag{A1}$$

$$E^1_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad E^2_{ijkl} = \delta_{ij}\delta_{kl},$$

$$E^3_{ijkl} = \delta_{ij} n_k n_l, \quad E^4_{ijkl} = n_i n_j \delta_{kl},$$

$$E^5_{ijkl} = \tfrac{1}{4}(n_i n_k \delta_{jl} + n_i n_l \delta_{jk} + n_j n_k \delta_{il} + n_j n_l \delta_{ik}),$$

$$E^6_{ijkl} = n_i n_j n_k n_l. \tag{A2}$$

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