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ELASTIC MEDIA WITH RANDOMLY DISTRIBUTED DEFFCTS

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Abstract

In this paper, the elastic field in a solid with randomly distributed defects is derived. These defects are composed of cavities and microcracks, whose locations, orientation and size are random variables. The Random Point Field Model is proposed to describe the random defects, and the basic equations for elastic field in a random defect medium are developed. Two examples are studied in detail. One is a solid with random microcracks and the other is a solid with ellipsoidal cavities.

I. Introduction

Engineering material contains a variety of defects, and these defects play a dominant role in material behaviour, particularly for brittle materials. Therefore, it is very important to study the effect of these defects in material. Budiansky etc[1] calculated the elastic moduli of a cracked solid using the self-consistent method. Whereas it is not enough to consider only the effect of microcracks on elastic moduli. the more essential problem may be how these defects influence the stress distribution in a material. The present paper attempts to study this problem by introducing the Random Point Field Model. The basic equations for elastic field in a random defect medium are developed. Two examples are studied in detail. One is a solid with random microcracks, and the other is a solid with ellipsoidal cavities.

II. Basic Theory

According to Ref [2] the basic equations of elastic field in an inhomogeneous material are

$$\varepsilon_{ij} = \varepsilon^0_{ij} - \int_V K_{ijkl}(\mathrm{x}-\mathrm{x}')C^1_{klmn}\varepsilon_{mn}dv(\mathrm{x}') \qquad (2.1)$$

$$\sigma_{ij}=\sigma^0_{ij} - \int_V S_{ijkl}(\mathrm{x}-\mathrm{x}')B^1_{klmn}\sigma_{mn}dv(\mathrm{x}') \qquad (2.2)$$

where

$$K_{ijkl}=-\frac{1}{4}(\partial_i\partial_l G_{jk}+\partial_j\partial_l G_{ik}+\partial_i\partial_k G_{jl}+\partial_j\partial_k G_{il}) \qquad (2.3)$$

$$S_{ijkl}=C^M_{ijkl}\delta(\mathrm{x}-\mathrm{x}')-C^M_{ijmn}K_{mnpq}C^M_{pqkl} \qquad (2.4)$$

$$C^1_{klmn}=C_{klmn}(\mathrm{x}')-C^M_{klmn} \qquad (2.5)$$

$$B^1_{klmn}=B_{klmn}(\mathrm{x}')-B^M_{klmn} \qquad (2.6)$$

where G_{ij} is the Green's function for an unbounded medium. $G_{ijmn}{}^{M}$ and $B_{ijmn}{}^{M}$ are elastic modulus tensor and compliance tensor of matrix, respectively. G_{klmn} and B_{klmn} are elastic modulus tensor and compliance tensor of the overall material, respectively ε_{ij}^{0} and σ_{ij}^{0} are the homogeneous solutions of matrix. For a random inclusion medium, we can obtain

$$C_{klmn}^{1}=\sum_{a=1}^{N_1} C_{klmn}^{a} V_a(\mathrm{x}') \tag{2.7}$$

$$B_{klmn}^{1}=\sum_{a=1}^{N_1} B_{klmn}^{a} V_a(\mathrm{x}') \tag{2.8}$$

where

$$V_a(\mathrm{x}')=\begin{cases}1 & (\mathrm{x}'\in h_a)\\ 0 & (\mathrm{x}'\overline{\in} h_a)\end{cases} \tag{2.9}$$

$$C_{klmn}^{a}=C_{klmn}^{I}-C_{klmn}^{M},\quad B_{klmn}^{a}=B_{klmn}^{I}-B_{klmn}^{M} \tag{2.10}$$

where h_a is the region of the a-th inclusion. $G_{klmn}{}^{I}$ and $B_{klmn}{}^{I}$ are elastic modulus tensor and compliance tensor of inclusion.

When the inclusions become cavities, one obtains

$$C_{klmn}^{I}=0,\qquad B_{klmn}^{I}\to\infty \tag{2.11}$$

Substituting equations (2.7), (2.8) and (2.11) into equations (2.1) and (2.2) gives

$$\varepsilon_{ij}=\varepsilon_{ij}^{0}+\sum_{a=1}^{N_V}\int_{h_a} K_{ijkl}(\mathrm{x}-\mathrm{x}')C_{klmn}^{M}\varepsilon_{mn}^{I}dv\ (\mathrm{x}') \tag{2.12}$$

$$\sigma_{ij}=\sigma_{ij}^{0}-\sum_{a=1}^{N_V}\int_{h_a} S_{ijkl}(\mathrm{x}-\mathrm{x}')\varepsilon_{kl}^{I}dv\ (\mathrm{x}') \tag{2.13}$$

Since the location, orientation and size of inclusion are random variables, the strain and stress determined by equations (2.12) and (2.13) are random field variables.

In what follows, we will consider the limiting transition from the cavity to a crack: considering a flattened cavity, which occupies a finite simply-connected region V with a smooth boundary; then cutting along a smooth oriented surface Ω which is bounded by a closed contour Γ which lies on the boundary of V. (Fig. 1).

Let us choose a local coordinate system at the point x', $\mathrm{x}^L(x_1^L, x_2^L, x_3^L)$, such that its x_3^L-axis is directed along the normal $\mathrm{n}(\mathrm{x}')$ Let $\epsilon(x)$ be the transverse dimension of the cavity and x_{31}^L, x_{32}^L be the coordinates of the points of intersection of the x_3^L-axis with the boundary of V, while $x_{31}^L, x_{32}^L\to 0$, as ϵ 0. For a fixed point $x\notin v$, the kernels K_{ijkl}, S_{ijkl} are smooth, bounded functions, hence,

$$\int_{h_a} K_{ijkl}(\mathrm{x}-\mathrm{x}')C_{klmn}^{M}\varepsilon_{mn}^{I}dV(\mathrm{x}')$$

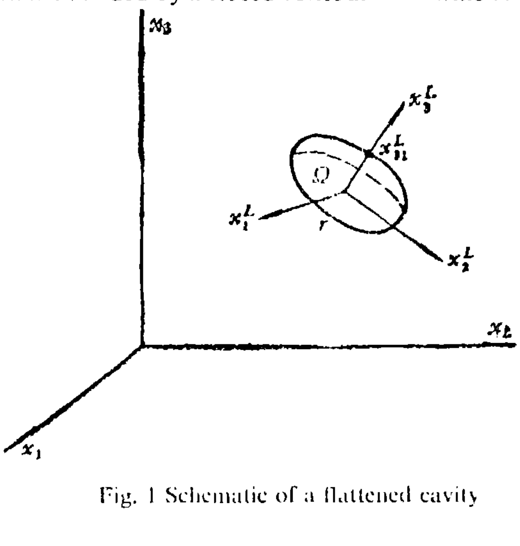

Fig. 1 Schematic of a flattened cavity

$$=\int_{\Omega_a} K_{ijkl}(\mathbf{x}-\mathbf{x}')C^M_{klmn}n_{(m}b_{n)}dS(\mathbf{x}')+O(\epsilon) \tag{2.14}$$

$$\int_{h_a} S_{ijkl}(\mathbf{x}-\mathbf{x}')\varepsilon^*_{kl}dV(\mathbf{x}')=\int_{\Omega_a} S_{ijkl}(\mathbf{x}-\mathbf{x}')n_{(k}b_{l)}dS(\mathbf{x}')+O(\epsilon) \tag{2.15}$$

where Ω_a is the a-th crack surface, and n_m, b_n are the unit normal vector of crack surface and the opening of the crack, respectively.

$$n_{(k}b_{l)}=\frac{1}{2}(n_k b_l+n_l b_k) \tag{2.16}$$

Substituting equations (2.14) and (2.15) into equations (2.12) and (2.13) yields the basic equations of elastic field for a cracked solid,

$$\varepsilon_{ij}=\varepsilon^0_{ij}+\sum_{a=1}^{N_V}\int_{\Omega_a} K_{ijkl}(\mathbf{x}-\mathbf{x}')C^M_{klmn}n_{(m}b_{n)}dS(\mathbf{x}') \tag{2.17}$$

$$\sigma_{ij}=\sigma^0_{ij}-\sum_{a=1}^{N_V}\int_{\Omega_a} S_{ijkl}(\mathbf{x}-\mathbf{x}')n_{(k}b_{l)}dS(\mathbf{x}') \tag{2.18}$$

Assumptions:

(i) In volume V, the number of defects obeys Poisson distribution with parameter λ, i.e., for $m=0,1,2,\cdots$

$$P_r[N_V=m]=(m!)^{-1}\left(\int_V \lambda dV\right)^m\cdot\exp\left[-\int_V \lambda dV\right] \tag{2.19}$$

where $P_r[N_v=m]$ is the probability of $N_v=m$, and λ means the mean unmber of defects in unit volume.

(2) $\{N_v;\ v\subset V\}$ has independent increments in regions with a restriction for the intersection of them, i.e., for $v=v_1,v_2,\cdots,v_k$,

$$P_r[N_{v_1}=n_1,\ N_{v_2}=n_2,\cdots,\ N_{v_k}=n_k]=\prod_{i=1}^{k}P_r[N_{v_i}=n_i] \tag{2.20}$$

According to equations (2.12), (2.13), (2.17) and (2.18), the field perturbations created by defects can be represented as

$$A_{ij}(\mathbf{x})=\sum_{a=1}^{N_V}A^a_{ij}(\mathbf{x}-\mathbf{r}_a;\ \phi_a) \tag{2.21}$$

where $\mathbf{r}_a$ is the center of the a-th defect. ϕ_a represents the orientation and size of the a-th defect. The characteristic function is defined as

$$M_A\triangleq E[\exp(i\alpha_{ij}A_{ij})] \tag{2.22}$$

where α_{ij} is a constant tensor, and the symbol $E[\ ]$ denotes the average, i is the imaginary unit. Substituting of equation (2.21) into equation (2.22) yields

$$M_A=E\left\{\exp\left[ia_{ij}\sum_{a=1}^{N_1}A^a_{ij}(\mathbf{x}-\mathbf{r}_a;\ \phi_a)\right]\right\} \tag{2.23}$$

By using the property of conditional expectation, equation (2.23) becomes

$$M_A=P_r[N_V=0]+\sum_{m=1}^{\infty}P_r[N_V=m]\cdot E\left\{\exp\left(i\sum_{a=1}^{m}a_{ij}A^a_{ij}\right)\Big|\left(N_V=m\right)\right\} \tag{2.24}$$

where E[ ] means the average only with respect to defect orientation and size under the condition that $N_v=m$. Substituting equations (2.19), (2.20) into equation (2.24) gives

$$M_A=\exp\left\{\int_V \lambda E[\exp(ia_{ij}A^a_{ij})-1]dV(\mathbf{r}_a)\right\} \tag{2.25}$$

By using the characteristic function, the average of A_{ij} is obtained as

$$\langle A_{ij}\rangle=i^{-1}\partial M_A/\partial a_{ij} \qquad \text{when } a_{ij}=0 \tag{2.26}$$

Substituting of equation (2.25) into equation (2.26) yields

$$\langle A_{ij}\rangle=\int_V \lambda E[A^a_{ij}(\mathbf{x}-\mathbf{r}_a;\ \phi_a)]dV(\mathbf{r}_a) \tag{2.27}$$

where the symbol E[ ] in the integrand means the average only with respect to the orientation and size of defects.

III. An Elastic Body with Randomly Distributed Ellipsoidal Cavities

By taking the last term in equations (2.12) and (2.13) as the tensor A, and substituting them into equation (2.27), the average equations can be obtained as

$$\langle\varepsilon_{ij}\rangle=\varepsilon^0_{ij}+\int_V \lambda E\left[\int_{h_a} K_{ijkl}(\mathbf{x}-\mathbf{x}')C^M_{klmn}\varepsilon^I_{mn}dV(\mathbf{x}')\right]dV(\mathbf{r}_a) \tag{3.1}$$

$$\langle\sigma_{ij}\rangle=\sigma^0_{ij}-\int_V \lambda E\left[\int_{h_a} S_{ijkl}(\mathbf{x}-\mathbf{x}')\varepsilon^I_{kl}dV(\mathbf{x}')\right]dV(\mathbf{r}_a) \tag{3.2}$$

If the cavities are uniformly distributed, λ equals a constant n, which means the average number of cavities in unit volume.

(1) The average strain within a cavity

Given that x is within the cavity, the intensity function λ is

$$\lambda(\mathbf{r}_a)=\begin{cases}\delta(\mathbf{r}_a-\mathbf{x}) & (\mathbf{r}_a\in h_\mathbf{x})\\ n & (\mathbf{r}_a\notin h_\mathbf{x})\end{cases} \tag{3.3}$$

where $h_\mathbf{x}$ is the region occupied by a cavity with its center at $\mathbf{x}$. Substituting equation (3.3) into equation (3.1) gives

$$E(\varepsilon^I_{ij})=\frac{1}{1-v_f}(\underset{\sim}{I}-\langle\underset{\sim}{P}\rangle:\underset{\sim}{C}^M)^{-1}_{ijkl}\varepsilon^0_{kl} \tag{3.4}$$

where I is the identity tensor, and v_f is the volume fraction of cavities. The tensor $\underset{\sim}{P}$, which was introduced by Kunin[3] in studying a single inclusion problem, is shown in Appendix I. < > denotes the average with respect to the orientation of ellipsoidal cavities.

(2) The stress concentration around an ellipsoidal cavity

By the aid of results obtained from the investigation of a single inclusion problem, the stress concentration is derived as

$$\sigma^{*}_{ij}(\mathbf{n})=S^{*}_{ijkl}(\mathbf{n})E(\epsilon^{*}_{kl}) \tag{3.5}$$

where n is the unit normal vector of the cavity surface

$$S^{*}_{ijkl}(\mathbf{n})=\frac{2\mu_0}{1-\gamma_0}[\gamma_0(E^{1}_{ijkl}-E^{2}_{ijkl}-E^{3}_{ijkl}) + (1-\gamma_0)(E^{4}_{ijkl}-2E^{5}_{ijkl})+E^{6}_{ijkl}] \tag{3.6}$$

where $E^{1}_{ijkl}\sim E^{6}_{ijkl}$, which depend on the Kronecker delta and on a unit vector, are shown in Appendix II, and μ_0 and γ_0 are shear modulus and Poisson ratio of the matrix.

(3) Examples

By using equations (3.4) and (3.5), the stress concentration around and ellipsoidal cavity in a solid with unidirectional cavities, or with random orientation cavities is calculated, respectively. The property of matrix is: $E_0=2.76$GPa, $\gamma_0=0.35$, the aspect ratio $\alpha=0.1$, 1, 10. The results for unidirectional cavities are shown in Figs. 2 – 11, where $v_f=0$ means the solution of a single cavity problem. From the results, it is found that the stress concentration is enhanced in the case of $v_f=0.4$. When the orientation of cavities is random, the result is shown in Table 1.

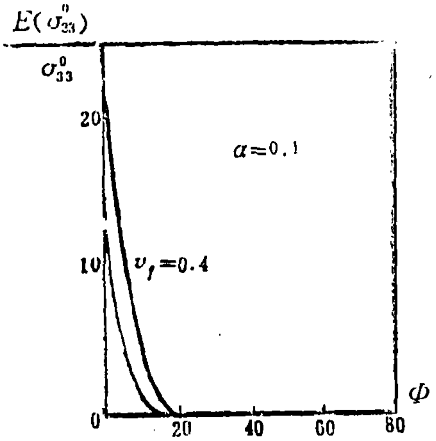

Fig. 2 The stress concentration around an unidirectional cavity

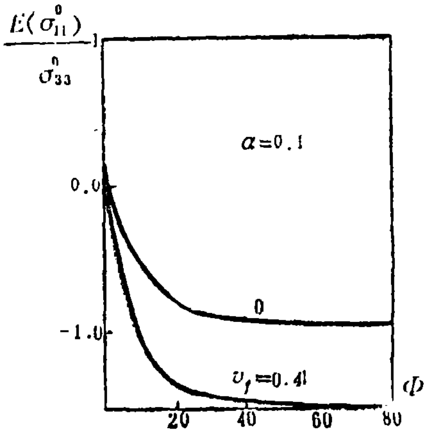

Fig.3. The stress concentration around an unidirectional cavity

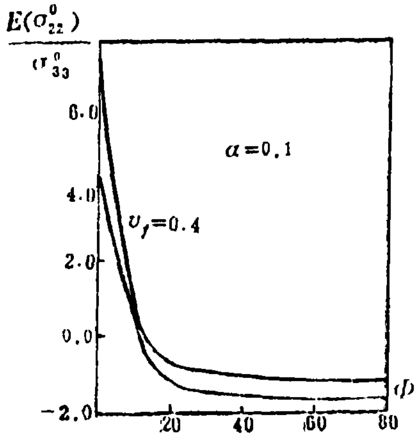

Fig. 4 The stress concentration around an unidirectional cavity

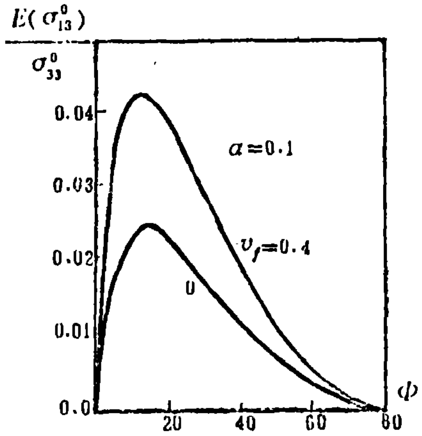

Fig. 5 The stress concentration around an unidirectional cavity

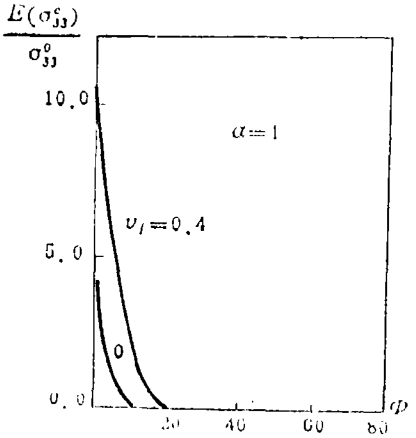

Fig. 6 The stress concentration around an unidirectional cavity

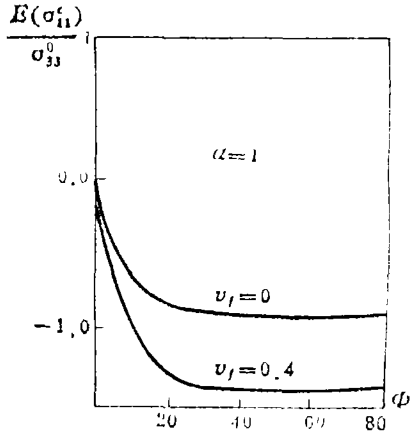

Fig. 7 The stress concentration around an unidirectional cavity

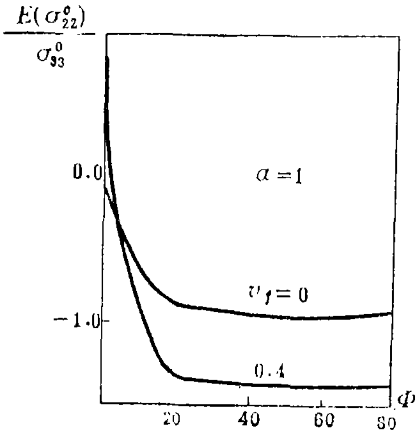

Fig. 8 The stress concentration around an unidirectional cavity

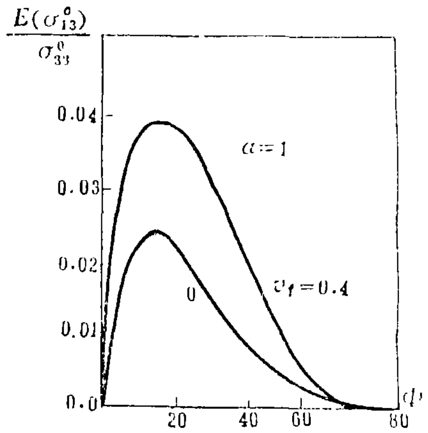

Fig. 9 The stress concentration around an unidirectional cavity

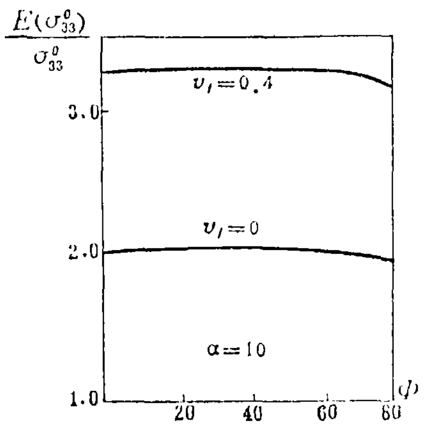

Fig. 10 The stress concentration around an unidirectional cavity

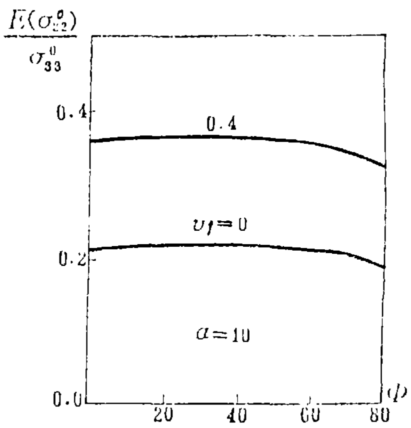

Fig. 11 The stress concentration around an unidirectional cavity

Table 1 The stress concentration around a random cavity

α		0.1				1.0				10			
v_f	Stress / Φ	$\frac{E(\sigma^e_{33})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{11})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{22})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{13})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{33})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{11})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{22})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{13})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{33})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{11})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{22})}{\sigma^0_{33}}$	$\frac{E(\sigma^e_{13})}{\sigma^0_{33}}$
0.0	0	2.07	0.00	0.21	0.00	2.07	0.00	0.21	0.00	2.07	0.00	0.21	0.00
	20	0.00	−0.78	−0.79	0.02	1.53	0.20	0.10	−0.56	2.07	0.00	0.21	−0.01
	40	0.00	−0.79	−0.79	0.00	0.52	0.37	−0.20	−0.44	2.07	0.00	0.21	−0.02
	60	0.00	−0.79	−0.79	0.00	0.02	−0.05	−0.54	0.03	2.07	0.00	0.21	0.04
	80	0.00	−0.79	−0.79	0.00	−0.02	−0.68	−0.76	0.12	2.06	0.00	0.21	−0.12
0.4	0	3.45	0.00	0.36	0.00	3.45	0.00	0.35	0.00	3.46	0.00	0.36	0.00
	20	0.00	−1.30	−1.31	0.04	2.56	0.34	0.16	−0.93	3.46	0.00	0.36	−0.01
	40	0.00	−1.31	−1.31	0.02	0.97	0.61	−0.33	−0.73	3.46	0.00	0.36	−0.03
	60	0.00	−1.31	−1.31	0.01	0.03	−0.09	−0.89	0.05	3.45	0.00	0.36	−0.06
	80	0.00	−1.31	−1.31	0.00	−.004	−1.13	−1.26	0.20	3.43	0.01	0.35	−0.20
0.8	0	10.40	0.00	1.07	0.00	0.40	0.00	1.07	0.00	10.40	0.00	1.07	0.00
	20	0.04	−3.91	−3.91	0.11	7.67	1.02	0.49	−2.79	10.40	0.00	1.07	−0.04
	40	0.00	−3.93	−3.93	0.05	2.61	1.84	−0.99	−2.19	10.40	0.00	0.07	−0.09
	60	0.00	−3.93	−3.93	0.02	0.09	−0.27	−2.69	0.16	10.40	0.00	1.07	−0.18
	80	0.00	−3.93	−3.93	0.01	−0.11	−3.39	−3.78	0.60	10.30	0.03	1.06	−0.68

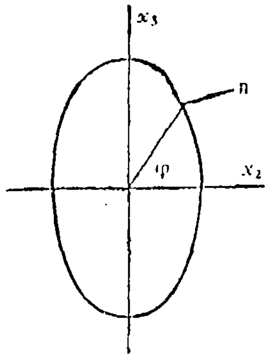

Fig. 12 Schematic of an ellipsoidal cavity

IV. An Elastic Body with Randomly Distributed Penney Shaped Cracks

By taking the last term in equations (2.17), (2.18) as the tensor A, and substituting them into equation (2.27), the equations for average strain and stress in a cracked solid are obtained as

$$\langle\varepsilon_{ij}\rangle=\varepsilon^{0}_{ij}+\int_{V}\lambda E\left[\iint_{\Omega_{a}}K_{ijkl}(\mathbf{x}-\mathbf{x}')C^{M}_{klmn}n_{(m}b_{n)}dS(\mathbf{x}')\right]dV(\mathbf{r}_{a}) \qquad (4.1)$$

$$\langle\sigma_{ij}\rangle=\sigma^{0}_{ij}-\int_{V}\lambda E\left[\iint_{\Omega_{a}}S_{ijkl}(\mathbf{x}-\mathbf{x}')n_{(k}b_{l)}dS(\mathbf{x}')\right]dV(\mathbf{r}_{a}) \qquad (4.2)$$

Assumption: If the average stress in matrix is $\langle\sigma^{M}_{pq}\rangle$, every crack is independent, and loaded by $\langle\sigma^{M}_{pq}\rangle$. By using the assumption, the opening of a crack can be obtained through the solution of an infinity containing a single crack. For simplicity, it is taken in the form as

$$\langle n_{(m}b_{n)}\rangle=\frac{1}{S_{0}}\int_{\Omega}\frac{1}{2}(n_{m}b_{n}+n_{n}b_{m})dS \qquad (4.3)$$

where S_0 is the area of the crack surface. In local coordinate system connected to the crack, one obtains

$$\langle n_{(m}b_{n)}\rangle=F^{L}_{mnpq}\langle\sigma^{M}_{pq}\rangle \qquad (4.4)$$

where

$$F^{L}_{3333}=\frac{16(1-\nu_{0}^{2})}{3E_{0}S_{0}}a^{3},\quad F^{L}_{1313}=F^{L}_{2323}=\frac{8(1-\nu_{0}^{2})}{3E_{0}(2-\nu_{0})S_{0}}\cdot a^{3} \qquad (4.5)$$

The other components are zero and a is the crack radius.

(1) The average stress in the matrix

Assume that x is in the matrix, which means that there is no crack in a neighborhood of $\mathbf{x}$, i.e.

$$\lambda(\mathbf{r}_{a})=\begin{cases}0 & (|\mathbf{r}_{a}-\mathbf{x}|\leqslant a)\\ n & \text{others}\end{cases} \qquad (4.6)$$

Substituting equation (4.6) into equation (4.2) yields

$$\langle\sigma^M_{ij}\rangle=[I-nS_0D:\langle F\rangle]^{-1}_{ijkl}\sigma^0_{kl} \tag{4.7}$$

where

$$D_{pqmn}=D_1E^1_{pqmn}+D_2E^2_{pqmn} \tag{4.8}$$

where

$$D_1=\frac{2\mu_0}{15}\times\frac{7-5\gamma_0}{1-\gamma_0}\ ,\ D_2=\frac{2\mu_0}{15}\times\frac{5\gamma_0+1}{1-\gamma_0} \tag{4.9}$$

a. The orientation of cracks is random.

In such a case, the average of tensor F is

$$\langle F_{ijkl}\rangle=F_1E^1_{ijkl}+F_2E^2_{ijkl} \tag{4.10}$$

where

$$F_1=\frac{32(1-\gamma_0^2)(5-\gamma_0)}{45(2-\gamma_0)S_0E_0}\cdot\langle a^3\rangle \tag{4.11}$$

$$F_2=-\frac{16\gamma_0(1-\gamma_0^2)}{45(2-\gamma_0)S_0E_0}\cdot\langle a^3\rangle \tag{4.12}$$

Substituting equations (4.8) and (4.10) into equation (4.7) gives

$$\langle\sigma^M_{ij}\rangle=(G_1E^1_{ijkl}+G_2E^2_{ijkl})\sigma^0_{kl} \tag{4.13}$$

where

$$G_1=\frac{1}{1-nS_0D_1F_1} \tag{4.14}$$

$$G_2=\frac{D_2F_1+D_1F_2+3D_2F_2}{(1-nS_0D_1F_1)^2+3(1-nS_0D_1F_1)(D_2F_1+D_1F_2+3D_2F_2)} \tag{4.15}$$

Under a state of uniaxial loading σ^0_{33}, the average stress components in the matrix are shown in Figs. 13 and 14 versus $n\langle a^3\rangle$.

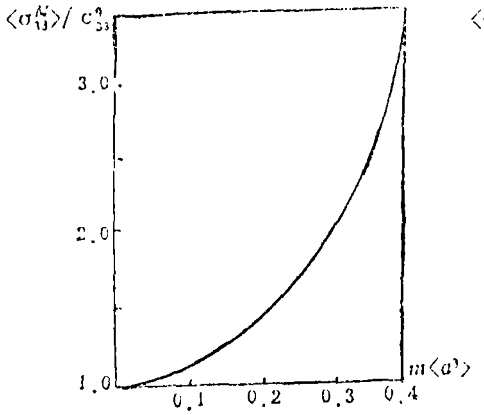

Fig. 13 The average stress in the matrix with microcracks

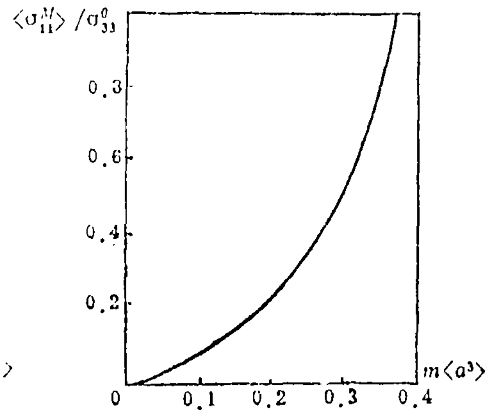

Fig. 14 The average stress in the matrix with microcracks

b. All the crack planes are parallel with the smae plane (e_1, e_2).

In such a case, substituting tensor F^L into equation (4.7) gives

$$E(\sigma^M_{33}) = \sigma^0_{33}/(1 + nS_0 D_{3333}\cdot F^L_3) \tag{4.16}$$

The other components can also be obtained easily.

(2) The elastic moduli of a cracked solid

It is assumed that the locations of microcracks are uniformely distributed, i.e.

$$\lambda = n \tag{4.17}$$

Substituting equation (4.17) into equation (4.1) yields

$$\langle \varepsilon_{ij}\rangle = \varepsilon^0_{ij} + nS_0 E(\langle n_{(i}b_{j)}\rangle) \tag{4.18}$$

where $E(\ )$ means the average with respect to the crack orientation. Through equation (4.18), the elastic compliance tensor can be expressed as

$$B^*_{ijkl} = B^0_{ijkl} + nS_0\langle F_{ijmn}\rangle\cdot[I - nS_0 D:\langle F\rangle]^{-1}_{mnkl} \tag{4.19}$$

a. The orientation of cracks is random.

Substituting equation (4.10) into equation (4.19) yields

$$\frac{\mu^*}{\mu_0} = \frac{1}{2\mu_0}\cdot\frac{1}{B^*_1}, \qquad \frac{E^*}{E_0} = \frac{1}{E_0}\cdot\frac{1}{B^*_1 + B^*_2} \tag{4.20}$$

where

$$B^*_1 = \frac{1}{2\mu_0} + nS_0 G_1, \quad B^*_2 = -\frac{\nu_0}{E_0} + nS_0(F_2G_1 + F_1G_2 + 3F_2G_2) \tag{4.21}$$

The results are shown in Fig. 15 and Fig. 16, where the doted line is the result of the self-consistent method[11], and the Young's moduli predicted by both the methods almost coincide with each other.

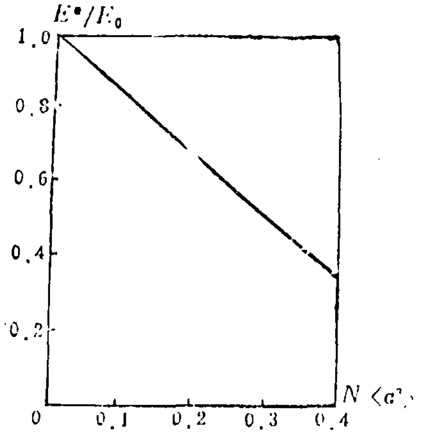

Fig. 15 The Young's modulus of a solid with random cracks

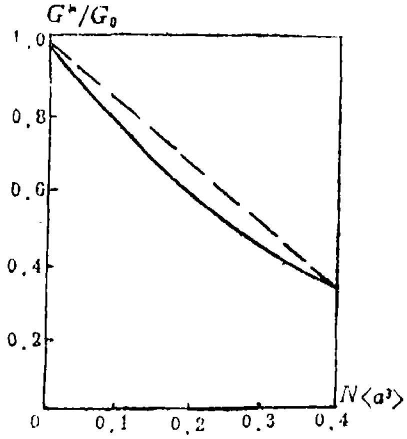

Fig. 16 The shear modulus of a solid with random cracks

b. All crack planes are parallel with (e_1, e_2).

In such a case, the overall material is quasi-isotropic, and

$$\frac{E^*_3}{E_0} = 1\Big/\left(1 + \frac{nS_0F^L_{3333}}{1 + nS_0 D_{3333}\cdot F^L_{3333}}\right) \tag{4.22}$$

$$E_1^* = E_2^* = E_0 \tag{4.23}$$

$$\frac{G_{13}^*}{G_0} = 1 \Big/ \left(1 + \frac{nS_0 F_{3333}^L}{1 + nS_0 D_{1313} \cdot F_{1313}^L}\right) \tag{4.24}$$

$$G_{12}^* = G_0 \tag{4.25}$$

Using the self-consistent method, Hoeing[1] calculated the elastic moduli numerically. The results obtained from both the methods are shown in Fig. 17 , where the doted line is the Hoeing's result.

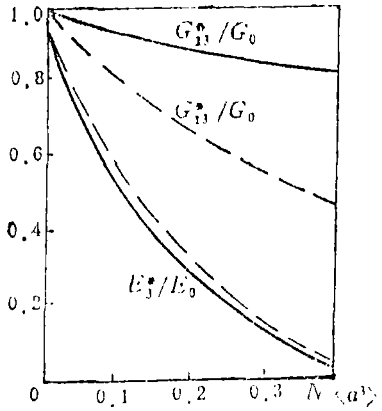

Fig. 17 Elastic moduli of a solid with unidirectional cracks

Acknowledgement

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Appendix I

In the coordinate system connected to the ellipsoid axes, the tensor P has the symmetry of the ellipsoid and is defined by nine essential components, i.e

$$P_{1111}^L = k_0(3I_{11} + (1 - 4\gamma_0)I_1], \quad P_{1122}^L = k_0(I_{21} - I_1)$$

$$P_{1212}^L = \frac{k_0}{2}[I_{21} + I_{12} + (1 - 2\gamma_0)(I_1 + I_2)] \tag{A1.1}$$

where

$$I_p = \frac{3}{2} v \int_0^\infty \frac{d\xi}{(a_p^2 + \xi)\Delta(\xi)}, \quad I_{pq} = \frac{3}{2} v a_p^2 \int_0^\infty \frac{d\xi}{(a_p^2 + \xi)(a_q^2 + \xi)\Delta(\xi)} \tag{A1.2}$$

$$k_0 = 1/[16\pi\mu_0(1 - \gamma_0)] \tag{A1.3}$$

$$\Delta(\xi) = \sqrt{(a_1^2 + \xi)(a_2^2 + \xi)(a_3^2 + \xi)} \qquad (p, q = 1, 2, 3) \tag{A1.4}$$

The remaining six tensor components are obtained by a cyclic replacement of the indices 1,2,3.

Appendix II

$$E_{ijkl}^1 = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$E_{ijkl}^2 = \delta_{ij}\delta_{kl}$$

$$E^3_{ijkl}=\delta_{ij}n_kn_l$$

$$E^4_{ijkl}=n_in_j\delta_{kl}$$

$$E^5_{ijkl}=\frac{1}{4}(n_in_k\delta_{jl}+n_in_l\delta_{kj}+n_jn_k\delta_{il}+n_jn_l\delta_{ik})$$

$$E^6_{ijkl}=n_in_jn_kn_l \tag{A2.1}$$

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