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Short English Theses

含随机裂纹场的弹性介质

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Elastic Medium with Randomly Distributed Cracks

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In this paper, a generalized theory is developed for calculating the bulk properties of an elastic medium with randomly distributed cracks. For the purpose of simplication, it is assumed that all cracks are opened, after the medium has been loaded. In such a case, the stress and strain fields in a cracked solid are represented in the form of:

$$\varepsilon(\vec{x}) = \varepsilon_0 + \int K_0(\vec{x} - \vec{x}')C_0 M(\vec{x}')\delta(\Omega)dV(\vec{x}') \tag{1}$$

and

$$\sigma(\vec{x}) = \sigma_0 - \int S_0(\vec{x} - \vec{x}')M(\vec{x}')\delta(\Omega)dV(\vec{x}') \tag{2}$$

where,

$$M(\vec{x}')\delta(\Omega) = \sum_k n_k\, b_k(\vec{x}')\delta(n_k) \tag{3}$$

In (3), n_k is the normal to the surface Ω_k, b_k^i is a vector filed at Ω_k, which can be interpreted as a jump of displacement across a crack, $\delta(\Omega_k)$ is a delta-function concentrated at Ω_k.

$$\begin{aligned} K^0{}_{ijkl} &= -[\partial_k \partial_l G_{ij}(\vec{x}, \vec{x}')]_{(ik)(jl)} \\ S^0{}_{ijkl} &= C^0{}_{ijkl} - C^0{}_{ijpq} \cdot K_0^{pqmn} \cdot C^0{}_{mnkl} \end{aligned} \tag{4}$$

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Where, G_{ij} is the tensor of Green's function for displacement, and C^0_{ijkl} is the elastic moduli tensor of a homogeneous matrix. Then the region Ω of all surfaces of the cracks and the region Ω_{x_0} are introduced and defined as:

$$\Omega_{x_0} = \begin{cases} \Omega = \bigcup\limits_i \Omega_i & if \quad x_0 \overline{\in} \Omega \\ \bigcup\limits_{j \neq i} \Omega_j & if \quad x_0 \in \Omega_i \end{cases} \tag{5}$$

Where, Ω_j is the region occupied by the j-th crack, and $\overline{\Omega}_{x_0}$ denotes the complement of Ω_{x_0} to Ω. It is immediately obvious that $\overline{\Omega}_{x_0}$ is the region in which the the point x_0 is situated. Hence the equations governing the stress and strain fields in the region of a cracked solid can be represented in the form of:

$$\varepsilon(\vec{x}) = \varepsilon_0 + \int K_0(\vec{x} - \vec{x}')C_0 n(\vec{x}')b(\vec{x}')\delta(\Omega)dV(\vec{x}')$$

$$\sigma(\vec{x}) = \sigma_0 - \int S_0(\vec{x} - \vec{x}')n(\vec{x}')b(\vec{x}')\delta(\Omega)dV(\vec{x}') \tag{6}$$

Where, $n(\vec{x}')b(\vec{x}')$ is an arbitrarily continuous tensor field coinciding with $n_k(\vec{x})b_k(\vec{x})$ on the surface Ω_k.

The mean equations of the stress and strain fields in a cracked solid are derived in the form of:

$$\langle\varepsilon(\vec{x})\rangle = \varepsilon_0 + \int K_0(\vec{x} - \vec{x}')C_0\langle\delta(\Omega)\rangle \cdot \langle n(\vec{x}')b(\vec{x}') \mid \vec{x}'\rangle\, dV(\vec{x}')$$

$$\langle\sigma(\vec{x})\rangle = \sigma_0 - \int S_0(\vec{x} - \vec{x}')\langle\delta(\Omega)\rangle \cdot \langle n(\vec{x}')b(\vec{x}') \mid \vec{x}'\rangle dV(\vec{x}') \tag{7}$$

As a result, the tensor of the effective compliance B^* can be represented in the form of:

$$B^* = B_0 - \frac{2}{3}\pi \times \frac{\langle a^3\rangle}{V_0} \cdot \langle nb \mid \vec{x}'\rangle \cdot \sigma_0^{-1} \tag{8}$$

Where, V_0 is the mean volume per crack, and a is the stochastic radius of circular crack in an isotropic medium.

Using the group expansion technique, we have derived the formula for b in the form of:

$$b(A_k) = \sum_{k=0}^{\infty} \sum_{A_k \in \mu_k(A)} b_k(A_k) \tag{9}$$

Where,

$$b_k(A_k) = b(A_k) - \sum_{i=0}^{\infty} \sum_{A_i \in \mu_i(A_k)} b_i(A_i) \tag{10}$$

$A = \{\vec{r}_1, \vec{r}_2, \cdots\}$ is the set of the position vectors of the crack centers. A subset of A containing k members is symbolized as A_k, and the set of all such subsets is symbolized as $\mu_k(A)$. Therefore the approximate solution of an infinity with infinite inclusions can be obtained, and the ensemble average expression for $\langle b|\vec{x'}\rangle$ is

$$\langle b|\vec{x}\rangle = \sum_{k=0}^{\infty}\int b_k(A_k)\cdot p(A_k|\vec{x'})dA_k \qquad (11)$$

Where, $p(A_k|\vec{x'})$ is a conditional probability density function for the configuration A_k on condition that the reference crack is lying at $\vec{x'}$ with its center.

As the first approximation, the result is obtained in the form of:

$$B^* = B_0 - \frac{8}{3}\cdot\frac{1-\gamma^0}{u_0(a-\gamma_0)}\cdot\frac{\langle a^3\rangle}{V_0}\cdot\langle aE^5(n)-\gamma_0E^6(n)\rangle$$

$$E^5{}_{\alpha\beta\lambda\mu} = \frac{1}{4}(n_\alpha n_\lambda\delta_{\beta\mu}+n_\alpha n_\mu\delta_{\beta\lambda}+n_\beta n_\lambda\delta_{\alpha u}+n_\beta n_\mu\delta_{\alpha\lambda})$$

$$E^6{}_{\alpha\beta\lambda\mu} = n_\alpha n_\beta n_\lambda n_\mu \qquad (12)$$

Here n_k is the normal to the surface Ω_k. The average values of E^5 E^6 can be given with respect to all possible orientation of cracks. If the distribution with respect to orientation is homogeneous, then

$$B^* = B_0 - \frac{8}{45}\cdot\frac{1-\gamma_0}{u_0(a-\gamma_0)}\cdot\frac{\langle a^3\rangle}{V_0}[\gamma_0E^2-2(5-\gamma_0)E^1] \qquad (13)$$

Where,

$$E^1{}_{\alpha\beta\lambda\mu} = \frac{1}{2}(\delta_{\alpha\lambda}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\lambda}),\ E^2{}_{\alpha\beta\lambda\mu} = \delta_{\alpha\beta}\delta_{\lambda\mu} \qquad (14)$$

In particular, for the effective shear modulus and the Poisson coefficient, we have

$$u^* = u_0\left[1+\frac{32}{45}\cdot\frac{\langle a^3\rangle}{V_0}\cdot\frac{(1-\gamma_0)(5-\gamma_0)}{(a-\gamma_0)}\right]^{-1} \qquad (15)$$

$$\frac{\gamma^*}{1+\gamma^*} = \frac{\gamma_0}{1+\gamma_0}\cdot\frac{u^*}{u_0}\left[1+\frac{16}{15}\cdot\frac{\langle a^3\rangle}{V_0}\cdot\frac{1-\gamma_0^2}{2-\gamma_0}\right] \qquad (16)$$